Solutions 2015 AB Exam

Texas A&M High School Math Contest

October 24, 2015

1. For how many integers n in the set $\{1, 2, 3, ..., 150\}$ is $n^3 - n^2$ the square of an integer?

   ANSWER: 13

   Solution: $n^3 - n^2 = n^2(n-1)$, so $n-1$ needs to be the square of an integer. For $1 \leq n \leq 150$ we obtain $0^2 = 1 - 1$
   $1^2 = 2 - 1$
   $2^2 = 5 - 1$
   $\vdots$
   $12^2 = 145 - 1$.
   So 13 in all.

2. If $\frac{1}{x} + \frac{1}{y} = 5$ and $\frac{1}{x} - \frac{1}{y} = 1$, find $x + y$.

   ANSWER: $\frac{5}{6}$

   Solution:
   $\frac{1}{x} + \frac{1}{y} = 5$
   $\frac{1}{x} - \frac{1}{y} = 1$, adding gives
   $\frac{2}{x} = 6$ and $x = \frac{1}{3}$
   Substitution gives
   $3 + \frac{1}{y} = 5$
   $\frac{1}{y} = 2$ and $y = \frac{1}{2}$
   $x + y = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$

3. Daisy has a bag containing 6 distinct objects. She draws one object and replaces it. She does this four times. What is the probability that she draws the same object exactly three times.

   ANSWER: $\frac{5}{54}$

   Solution: There are 6^4 possible outcomes. We need to count the number of outcomes having the same object exactly three times. There are 6 distinct objects, and the number of ways that object A can be picked three times is $\binom{4}{3} \cdot 5 = 4 \cdot 5 = 20$. So the probability is
   $\frac{6 \cdot (4 \cdot 5)}{6^4} = \frac{20}{6^3} = \frac{4 \cdot 5}{4 \cdot 9 \cdot 6} = \frac{5}{54}$

4. The sum of the first ten terms of a nonzero geometric series is 244 times the sum of the first five terms. What is the common ratio of this series?

   ANSWER: 3

   Solution:
   $a + ar + ... + ar^9 = \left(a + ar + ar^2 + ar^3 + ar^4\right)\left(1 + r^5\right) = 244(a + ar + ar^2 + ar^3 + ar^4)$
   ]5pt] Thus, $1 + r^5 = 244$ or $r^4 = 243$.

   Hence $r = 3$.

5. Write 2015 in the base 5 numeration system.

   ANSWER: 31030

   Solution:

$5^1 = 5$	$2015 = 3(625) + 140$
$5^2 = 25$	$140 = 1(125) + 15$
$5^3 = 125$	$15 = 3(5)$
$5^4 = 625$	

   So, $2015_{10} = 31030_5$

6. How many members of the set $\{\frac{1}{7}, \frac{2}{7}, \frac{3}{7}, ..., \frac{100}{7}\}$ are integer multiples of $\frac{2}{5}$?

   ANSWER: 7

   Solution: Suppose $\frac{K}{7}$ is an integer multiple of $\frac{2}{5}$. Then $\frac{K}{7} = n\frac{2}{5}$ for some integer n.

$$5K = 14n$$

   So 14 divides $5K$ and since it is relatively prime to 5, 14 divides K. Moreover, $100 = 7\times14+2$. Thus, there are 7 integers between 1 and 100 divisible by 14.

7. If f is a function such that $f(1) = 1$ and $f(n) = f(n-1) + 2n - 1$ for all $n \geq 2$, find $f(100)$.

ANSWER: $10,000$

Solution:

$$\begin{aligned}
f(100) &= f(99) + 2(100) - 1 \\
&= f(98) + 2(99) - 1 + 2(100) - 1 \\
&= f(98) + 2(99 + 100) - 2 \\
&= f(97) + 2(98) - 1 + 2(99 + 100) - 2 \\
&= f(97) + 2(98 + 99 + 100) - 3 \\
&\ \ \vdots \\
&= f(1) + 2(2 + 3 + ... + 99 + 100) - 99 \\
&= 1 + 2(\tfrac{(100)(101)}{2} - 1) - 99 \\
&= (100)(101) - 100 \\
&= 10100 - 100 \\
&= 10000
\end{aligned}$$

8. Find the value of the number A such that the three solutions to $x^3 - 8x^2 + Ax - 12 = 0$ are positive integers.

ANSWER: 19

Solution: Let α, β, γ be the three positive integer solutions to $x^3 - 8x^2 + Ax - 12 = 0$. Then $\alpha\beta\gamma = 12$ and $\alpha + \beta + \gamma = 8$. By inspection the three solutions are $\{1, 3, 4\}$. So

$$\begin{aligned}
A &= \alpha\beta + \beta\gamma + \alpha\gamma \\
&= 3 + 12 + 4 = 19
\end{aligned}$$

9. Let $P(x)$ be a monic polynomial of degree 3. (So the coefficient of x^3 is 1.) Suppose that the remainder when $P(x)$ is divided by $x^2 - 5x + 6$ equals 2 times the remainder when $P(x)$ is divided by $x^2 - 5x + 4$. If $P(0) = 100$, find $P(5)$.

ANSWER: 110

Solution: We are given

$$\begin{aligned} P(x) &= (x^2 - 5x + 6)(x + C_1) + 2(Ax + B) \\ &= (x^2 - 5x + 4)(x + C_2) + Ax + B \end{aligned}$$

where C_1, C_2, A, B are constants.

$$P(0) = 6C_1 + 2B = 4C_2 + B = 100.$$

So

$$\begin{aligned} P(5) = 6(5 + C_1) + 10A + 2B &= 30 + 6C_1 + 2B + 10A \\ &= 30 + 100 + 10A \\ &= 130 + 10A \end{aligned}$$

and

$$\begin{aligned} P(5) = 4(5 + C_2) + 5A + B &= 20 + 4C_2 + B + 5A \\ &= 20 + 100 + 5A \\ &= 120 + 5A. \end{aligned}$$

Hence

$$130 + 10A = 120 + 5A \implies 5A = -10 \implies A = -2\,.$$

Thus, $P(5) = 120 - 10 = 110$.

10. How many positive integers less than 1400 have no repeating digits, i.e. no digit occurs more than once. (For example 103 is such an integer, but 131 is not.)

ANSWER: 906

Solution:

(a) Every single digit (positive) integer is allowed. There are 9 of these.

(b) Two digits: $9 \times 9 = 81$

(c) Three digits: $9 \times 9 \times 8 = 648$

(d) Four digits: $1 \times 3 \times 8 \times 7 = 168$

Adding gives
$9 + 81 + 648 + 168 = 906$

11. Find all possible ordered pairs (A, B) of digits such that the integer $7A8B$ is divisible by 45.

ANSWER: $(3,0),(7,5)$

Solution: Since $7A8B$ is divisible by $45 = 9 \cdot 5$ it is divisible by 5 and $B \in \{0,5\}$. If $B = 0$ then $7 + A + 8 + 0$ must be divisible by 9 so $A = 3$.
If $B = 5$ then $7 + A + 8 + 5$ must be divisible by 9, i.e. $A = 7$. So

$$(A, B) \in \{(3,0),(7,5)\}$$

12. Among all real numbers x and y such that $|x+y| + |x-y| = 10$, what is the largest value of $x^2 + y^2 - 8y$?

ANSWER: 90

Solution: The graph of $|x+y| + |x-y| = 10$ is the square with vertices $(\pm 5, \pm 5)$. We have $x^2 + y^2 - 8y = x^2 + y^2 - 8y + 16 - 16 = x^2 + (y-4)^2 - 16$. So the maximum value occurs when x^2 and $(y-4)^2$ are maximal. This occurs (on the square) when $x = 5$ and $y = -5$. The maximum value is

$$\begin{aligned} 5^2 + (-9)^2 - 16 &= 25 + 81 - 16 \\ &= 106 - 16 = \underline{90} \end{aligned}$$

13. How many ways can one list $1,2,3,4$ so that no integer is followed by its successor, i.e. $n, n+1$ never occurs? For example $2,1,4,3$ is one such listing but $2,1,3,4$ is not (4 is the successor of 3).

ANSWER: 11.

Solution: 1 can only be followed by 3 or 4, 2 by 1 or 4, 3 by 1 or 2, 4 by 1, 2, or 3. The possible lists, of which there are 11, follow:

1324, 1432

2143, 2413, 2431

3142, 3214, 3241

4132, 4213, 4321

14. Exactly one of the five integers listed below is a prime. Which one is the prime number?

(a) $999,991$ (b) $999,973$ (c) $999,983$ (d) $1,000,001$ (e) $7,999,973$

ANSWER: (c) or $999,983$

Solution: A difference of two squares and the sum or difference of two cubes always factors.

$$\begin{aligned} 999,991 &= 1,000,000 - 9 = 1000^2 - 3^2 \\ 999,973 &= 100^3 - 3^3 \\ 1,000,001 &= 100^3 + 1^3 \\ 7,999,973 &= 200^3 - 3^3 \end{aligned}$$

So $999,983$ is the prime.

15. The set of integers $\{1, 2, ..., 20\}$ is divided into k disjoint subsets $S_1, S_2, ..., S_k$ such that no two different integers whose sum is divisible by 5 are in the same subset. For example 7 and 8 are not in the same subset since 5 divides $7 + 8$. What is the least possible value of k?

ANSWER: 4

Solution: The integers $5, 10, 15, 20$ must each be in a different subset. So $k \geq 4$. In fact, $k = 4$ using

$$\begin{aligned} S_1 &= \{1, 2, 5, 6, 7, 11, 12, 16, 17\} \\ S_2 &= \{3, 4, 8, 9, 10, 13, 14, 18, 19\} \\ S_3 &= \{15\} \\ S_4 &= \{20\} \end{aligned}$$

16. A number system with base 26 uses as its symbols for digits $A = 0, B = 1, C = 2, ..., Y = 24, Z = 25$. Express $P + Q$ in this system.

ANSWER: BF

Solution: $P = 15, Q = 16$. In base ten $P + Q$ is $15 + 16 = 31 = 1 \cdot 26^1 + 5 \cdot 26^0$ which is BF.

17. How many ordered pairs (x, y) of integers satisfy $x^2 + 6x + y^2 = 16$?

ANSWER: 12

Solution:

$$x^2 + 6x + y^2 = 16 \implies (x + 3)^2 + y^2 = 25$$

So 25 is the sum of two squares. The possible solutions of $z^2 + w^2 = 25$ are:

$$\begin{gathered}(0,5),(0,-5),(5,0),(-5,0)\\(3,4),(3,-4),(-3,4),(-3,-4)\\(4,3),(4,-3),(-4,3),(-4,-3)\,.\end{gathered}$$

Each of these solutions leads to a distinct solution of $x^2 + 6x + y^2 = 16$.

18. Recall that if S is a set then $|S|$ denotes the number of elements in S. Now let $S = \{1, 2, 3, 4, 5, 6\}$. How many different nonempty subsets T of S do not contain $|T|$? For example $T = \{3, 5\}$ is one such subset since $|T| = 2$ is not in T.

ANSWER: 31

Solution: The number of subsets of size 1 that do not contain 1 is $\binom{5}{1}$. The number of subsets of size 2 that do not contain 2 is $\binom{5}{2}$. The number of subsets of size k that do not contain k is $\binom{5}{k}$. So we have

$$\binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5} = 2^5 - 1 = 32 - 1 = 31$$

19. Find all positive integers n of the form $n = p^2q$ with p and q distinct primes and such that the sum of the reciprocals of all of the divisors of n is 2.

ANSWER: 28

Solution: The divisors of n are $1, p, p^2, q, pq, pq^2$ and we want

$$\tfrac{1}{1} + \tfrac{1}{p} + \tfrac{1}{p^2} + \tfrac{1}{q} + \tfrac{1}{pq} + \tfrac{1}{p^2q} = 2$$

i.e.

$$\begin{gathered}p^2q + pq + q + p^2 + p + 1 = 2p^2q\\(q + 1)(p^2 + p + 1) = 2p^2q.\end{gathered}$$

Note that $p^2 + p + 1$ is odd and must divide p^2q. Hence $p^2 + p + 1 = p, p^2, pq, p^2q$ or q. Since p does not divide $p^2 + p + 1$ then the only possibility is $p^2 + p + 1 = q$. This equation implies p divides $q - 1$. But

$$(q + 1)(p^2 + p + 1) = (q + 1)q = 2p^2q$$

so $q + 1 = 2p^2$. Hence p divides $q + 1$. Since p divides both $q - 1$ and $q + 1$ then $p = 2$. Also $q = p^2 + p + 1 = 4 + 2 + 1 = 7$ and $n = 4 \cdot 7 = 28$.

20. In Dr. Stecher's special topics class, 20% of the students are juniors and 80% are seniors. On a recent test the average score for the entire class was 85 and the average score for the seniors was 88. What was the average score for the juniors?

ANSWER: 73

Solution: We may assume there are 20 juniors with scores $A_1, ..., A_{20}$ and 80 seniors with scores $B_1, ..., B_{80}$. We have

$$\frac{B_1 + ... + B_{80}}{80} = 88, \text{ so } B_1 + ... + B_{80} = (88)(80)$$

and

$$\frac{A_1 + ... + A_{20} + B_1 + \cdots + B_{80}}{100} = 85$$

so

$$\begin{aligned} A_1 + ... + A_{20} &= (85)(100) - (B_1 + ... + B_{80}) = 85(100) - 88(80) \\ \frac{A_1 + ... + A_{20}}{20} &= \frac{85(100) - 88(80)}{20} \\ &= \frac{(85)(10) - 88(8)}{2} = 85(5) - 88(4) = 73 \end{aligned}$$

21. Let f_1 be the function defined by $f_1(x) = 1 - \frac{1}{x}$. Let f_2 be defined by $f_2(x) = f_1(f_1(x))$. Let f_3 be defined by $f_3(x) = f_1(f_2(x))$. Continue in this fashion to define the sequence of functions $f_1, f_2, ..., f_n, ...$ Find the value of $f_{50}(50)$.

ANSWER: $-\frac{1}{49}$

Solution:

$$\begin{aligned} f_1(50) &= 1 - \frac{1}{50} = \frac{49}{50} \\ f_2(50) &= f_1(\frac{49}{50}) = 1 - \frac{50}{49} = -\frac{1}{49} \\ f_3(50) &= f_1(-\frac{1}{49}) = 1 - \frac{1}{-\frac{1}{49}} = 50 \end{aligned}$$

From here on the values repeat: $f_4(50) = f_1(50) = \frac{49}{50}$ etc. So

$$f_{50}(50) \in \{\tfrac{49}{50}, -\tfrac{1}{49}, 50\}.$$

Since $50 = 48 + 2 = 3(16) + 2$ then

$$f_{50}(50) = f_2(50) = -\tfrac{1}{49}.$$