2015 Best Student Exam Solutions
Texas A&M High School Students Contest
October 24, 2015

1. It is known that $x+y=5$ and $x+y+x^2y+xy^2=24$. Find x^3+y^3.

   Answer: 68.

   Solution

$$x+y+x^2y+xy^2 = x+y+xy(x+y) = (x+y)(xy+1) = 5(xy+1) = 24.$$

   Therefore, $xy = \frac{24}{5} - 1 = \frac{19}{5}$.

   Therefore

$$x^3+y^3 = (x+y)(x^2-xy+y^2) = (x+y)((x+y)^2-3xy) = 5(5^2-\frac{57}{5}) = 5^3-57 = 125-57 = 68.$$

2. What is the minimal positive integer such that dividing it by 2 one gets a square of an integer, dividing it by 3 one gets a cube of an integer, and dividing it by 5 one gets the fifth power of an integer. Write you answer in terms of the prime decomposition of this number.

   Answer $2^{15}3^{10}5^6$.

   Solution. The prime number decomposition of a positive integer n can be represented in the following form:

$$n = 2^{\alpha_2}3^{\alpha_3}5^{\alpha_5}\ldots = \prod_{\text{all primes } p} p^{\alpha_p}$$

   where α_p are nonnegative integers and only finite number of α_p's are not equal to zero.

   The number $\frac{n}{2}$ is a square means that $\alpha_2 - 1$ and all other α_p are nonnegative even numbers. The number $\frac{n}{3}$ is a cube means that $\alpha_3 - 1$ and all other α_p are nonnegative numbers divisible by 3. The number $\frac{n}{5}$ is a fifth power of an integer means that $\alpha_5 - 1$ and all other α_p are nonnegative numbers divisible by 5. Consequently, we have that

   - α_2 is such that $\alpha_2 - 1$ is nonnegative even and α_2 is divisible by 3 and 5, i.e divisible by 15. Therefore, $\alpha_2 = 15$ is the minimal possible integer satisfying this property.
   - α_3 is such that $\alpha_3 - 1$ is nonnegative multiple of 3 and α_3 is divisible by 2 and 5, i.e divisible by 10. Therefore, $\alpha_3 = 10$ is the minimal possible integer satisfying this property.
   - α_5 is such that $\alpha_5 - 1$ is nonnegative multiple of 5 and α_5 is divisible by 2 and 3, i.e divisible by 6. Therefore, $\alpha_5 = 6$ is the minimal possible integer satisfying this property.
   - α_p for all primes p greater than 5 are nonnegative multiple of 2,3, and 5 and therefore $\alpha_p = 0$ is the minimal integer satisfying this property.

   Therefore, the minimal possible positive integer with required properties is equal to $2^{15}3^{10}5^6$.

3. α and β are two angles from the interval $[0, \frac{\pi}{2})$ such that the following two relations hold

$$\begin{cases} 3\sin^2\alpha + 2\sin^2\beta = 1, \\ 3\sin(2\alpha) - 2\sin(2\beta) = 0. \end{cases}$$

Find $\alpha + 2\beta$.

Answer $\dfrac{\pi}{2}$.

Solution. From the first equation $3\sin^2\alpha = 1 - 2\sin^2\beta = \cos(2\beta)$. From the second equation $\sin(2\beta) = \frac{3}{2}\sin(2\alpha)$.
Therefore,

$$\cos(\alpha + 2\beta) = \cos\alpha\cos(2\beta) - \sin\alpha\sin(2\beta) =$$
$$\cos\alpha \cdot 3\sin^2\alpha - \sin\alpha \cdot \frac{3}{2}\sin(2\alpha) = 3\cos\alpha\sin^2\alpha - 3\cos\alpha\sin^2\alpha = 0.$$

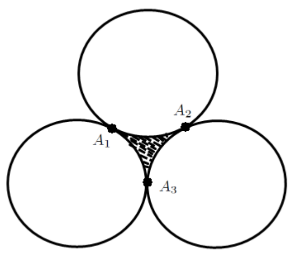
Since α and β belong the interval $[0, \frac{\pi}{2})$, we have that $\alpha+2\beta$ belong to the interval $[0, \frac{3\pi}{2})$. Therefore the fact that $\cos(\alpha + 2\beta) = 0$ implies that $\alpha + 2\beta = \dfrac{\pi}{2}$.

4. In a Chemistry class of 25 students the teacher randomly chooses two students to assist in an experiment. The probability that both chosen students are boys is equal to $\dfrac{3}{25}$. How many girls are in the Chemistry class?

Answer 16.

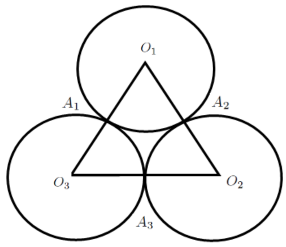
Solution Let n be the number of boys in the class. Then the number of order pairs of boys is equal to $n(n-1)$. The total number of ordered pairs is equal $25 \cdot 24$. Therefore the probability that two boys are chosen is equal to $\dfrac{n(n-1)}{25 \cdot 24}$ and from the assumption we get the following equation for n: $\dfrac{n(n-1)}{\cancel{25} \cdot 24} = \dfrac{3}{\cancel{25}}$. Equivalently, $n^2 - n - 72 = 0 \Leftrightarrow (n-9)(n+8) = 0$. Therefore $n = 9$ and the number of girls is 16.

5. Three circles of radius 1 are tangent one to each other in points A_1, A_2, A_3. Find the area of the curvilinear triangle with sides $\overset{\smile}{A_1A_2}$, $\overset{\smile}{A_2A_3}$, and $\overset{\smile}{A_3A_1}$ being the shorter arc on the corresponding circle, i.e. the area of the shape shaded in the figure.



Answer $\sqrt{3} - \frac{\pi}{2}$.

Solution Draw the triangle $O_1O_2O_3$ with vertices in the centers O_1, O_2, and O_3 of the circles (see the figure).



Then the required area is equal to the area of the equilateral triangle $\triangle O_1O_2O_3$ with the side of length 2 minus the areas of three sectors $O_1\ \overset{\smile}{A_1A_2}$, $O_2\ \overset{\smile}{A_2A_3}$, and $O_3\ \overset{\smile}{A_3A_1}$ of unit disk with the angle $\pi/3$. The area of the triangle $\triangle O_1O_2O_3$ is equal to $\sqrt{3}$ and the area of each of the three sectors is equal to $\frac{\pi}{6}$. Therefore, the area of the shaded shape is $\sqrt{3} - 3\frac{\pi}{6} = \sqrt{3} - \frac{\pi}{2}$.

6. 40 people vote for 4 candidates. Each voter can choose only one candidate, and in the voting results only the number of votes each candidate receives count. What is the number of all possible voting results? You may express your answer in terms of binomial coefficients.

   Answer $\binom{43}{3} = 43 \cdot 7 \cdot 41 = 12,341$.

   Solution Each result of voting is given by an (ordered) quadruple of nonnegative integer numbers (v_1, v_2, v_3, v_4) such that $v_1 + v_2 + v_3 + v_4 = 40$, where v_i is the number of votes for the ith candidate. We can equivalently encode this data by an ordered tuple consisting of 40 ones and 3 zeros: first put v_1 times 1, then put one time 0 (as a delimiter between the votes for the first candidate and the second candidate) then put v_2 times 1, then one time zero (as a delimiter between the votes for the second candidate and the first candidate) and so on. In other words, the number of results of the voting is equal to the number of order tuples of 40 ones and 3 zeros, i.e. $\binom{43}{3} = 43 \cdot 7 \cdot 41 = 12,341$.

7. $f(x)$ is defined for all real x, except $x = 1$, and satisfies the following equality:

$$(x-1)f\left(\frac{x+1}{x-1}\right) = 2x - 3f(x).$$

   Find $f(-1)$.

   Answer $-\dfrac{6}{7}$

   Solution Plugging $x = 0$ into the functional relation for f we get

$$-f(-1) = -3f(0). \tag{1}$$

   Plugging $x = -1$, we get

$$-2f(0) = -2 - 3f(-1) \tag{2}$$

   From (1) it follows that $f(0) = \dfrac{f(-1)}{3}$. Plugging this into (2) we get $-\dfrac{2f(-1)}{3} = -2 - 3f(-1)$. Therefore, $\dfrac{7f(-1)}{3} = -2 \Rightarrow f(-1) = -\dfrac{6}{7}$.

8. Given two parallel straight lines, mark 8 points on one line and 9 points on the other line. Draw the segments between each pair of points lying in two different lines. Assume that if three segments intersect in one point, then that point is an endpoint. What is the total number of points of intersection of all drawn segments that are not the endpoints of these segments? You may express your answer in terms of binomial coefficients.

   Answer:$\binom{8}{2}\binom{9}{2} = 28 \cdot 36 = 1008$.

   Solution By assumptions to each point of intersections of two segments that is not an endpoint of the segment one can assign exactly one quadrilateral with vertices being the endpoints of these two segments. This means that the required number of points is equal to the number of quadrilaterals with two vertices chosen from the 8 points marked on the first line and other two vertices chosen from 9 points marked on the second line. The latter number is equal to $\binom{8}{2}\binom{9}{2} = 28 \cdot 36 = 1008$.

9. Thirteen boys and d girls participate in math contest. The total number of points they earn is $d^2 + 10d + 17$ and it is known that all students get the same integer number of points. Find the largest possible number of participants in this contest.

   Answer:56.

   Solution The total number of participants is equal to $d + 13$ and by assumptions it divides $d^2 + 10d + 17$. Dividing the polynomial $p(x) = x^2 + 10x + 17$ by the polynomial $x + 13$ (using, for example, long division) we will get the following expression for $p(x)$:

$$x^2 + 10x + 17 = q(x)(x + 13) + r, \tag{3}$$

   where $r = 56$ and $q(x)$ is a polynomial of degree 1 (to find r plug for example $x = -13$ into (3) or proceed as follows:

$$x^2 + 10x + 17 = x^2 + 13x - 3x - 39 + 56 = (x - 3)(x + 13) + 56.)$$

   Since for $x = d$ the left hand-side of (3) is divisible by $d+13$ and the first term of the right hand-side of (3) is divisible by $d + 13$ then $r = 56$ must be divisible by $d + 13$ as well (which in turn is equal to the number of participants). Therefore the largest possible number of participants is equal to the maximal divisor of 56, i.e., to 56 itself.

10. Assume that $\frac{p}{q}$ is the irreducible fraction (the fraction in lowest terms) such that

$$\frac{p}{q} = \left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right)\left(1 - \frac{1}{16}\right)\dots\left(1 - \frac{1}{225}\right)$$

   Find $\frac{p}{q}$.

   Answer. $\frac{8}{15}$.

   Solution Note that

$$1 - \frac{1}{i^2} = \left(1 - \frac{1}{i}\right)\left(1 + \frac{1}{i}\right) = \frac{i-1}{i} \cdot \frac{i+1}{i}.$$

   Therefore

$$\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right)\cdots\left(1-\frac{1}{225}\right)=\left(\frac{1}{2}\cdot\frac{3}{2}\right)\left(\frac{2}{3}\cdot\frac{4}{3}\right)\cdots\left(\frac{14}{15}\cdot\frac{16}{15}\right)=$$

$$\left(\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{\cancel{3}}\cdot\ldots\cdot\frac{\cancel{14}}{15}\right)\left(\frac{\cancel{3}}{2}\cdot\frac{\cancel{4}}{\cancel{3}}\cdot\ldots\cdot\frac{16}{\cancel{15}}\right)=\frac{1}{15}\cdot\frac{16}{2}=\frac{8}{15}.$$

11. Assume that $f(x)=x^2+10x+20$. Find the largest real solution of the equation $f(f(f(f(x))))=0$.

Answer: $-5+5^{1/16}$

Solution Note that $f(x)=(x+5)^2-5$. Therefore,

$$f(f(x))=(f(x)+5)^2-5=((x+5)^2-\cancel{5}+\cancel{5})^2-5=(x+5)^4-5$$

In the same way,

$$f(f(f(x)))=(x+5)^8-5,\quad f(f(f(f(x))))=(x+5)^{16}-5.$$

Hence,

$$f(f(f(f(x))))=0\Leftrightarrow(x+5)^{16}=5\Leftrightarrow x+5=\pm5^{1/16}\Leftrightarrow x=-5\pm5^{1/16}$$

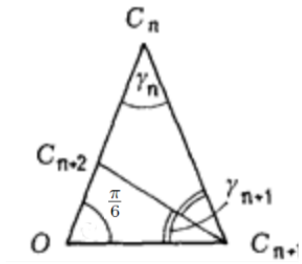
The largest solution is $x=-5+5^{1/16}$.

12. Consider a triangle C_1C_2O with $\angle C_1OC_2=\frac{\pi}{6}$. Let C_2C_3 be the bisector in the triangle C_1C_2O, then C_3C_4 be the bisector in the triangle C_2C_3O, and, generally, $C_{n+1}C_{n+2}O$ be the bisector in the triangle $C_nC_{n+1}O$ for any $n\geq 1$. Let $\gamma_n=\angle C_{n+1}C_nO$. Find the limit of γ_n as $n\to+\infty$.

Answer $\dfrac{5\pi}{18}$.

Solution. Since $C_{n+1}C_{n+2}$ is the bisector in the triangle $C_nC_{n+1}O$, we have that $2\gamma_{n+1}+\gamma_n+\dfrac{\pi}{6}=\pi$ or

$$2\gamma_{n+1}+\gamma_n=\frac{5\pi}{6}.\tag{4}$$



Therefore, if the limit of γ_n as $n\to+\infty$ exists and equal to γ, then passing to the limit in (4) as $n\to+\infty$ and using that $\lim\limits_{n\to+\infty}\gamma_{n+1}=\lim\limits_{n\to+\infty}\gamma_n=\gamma$ we get that $3\gamma=\dfrac{5\pi}{6}$, therefore $\gamma=\dfrac{5\pi}{18}$.

It remains to prove that $\lim\limits_{n\to+\infty} \gamma_n$ indeed exists (note that with the proof of existence below we automatically evaluate the limit γ so that the previous arguments for evaluation of γ can be omitted) From (4) it follows that

$$2\left(\gamma_{n+1} - \frac{5\pi}{18}\right) = \frac{5\pi}{18} - \gamma_n.$$

Therefore,

$$\left|\gamma_{n+1} - \frac{5\pi}{18}\right| = \frac{1}{2}\left|\gamma_n - \frac{5\pi}{18}\right|$$

Hence, recursively,

$$\left|\gamma_n - \frac{5\pi}{18}\right| = \frac{1}{2^{n-1}}\left|\gamma_1 - \frac{5\pi}{18}\right|,$$

Therefore, $\lim\limits_{n\to+\infty}\left|\gamma_n - \frac{5\pi}{18}\right| = 0$, which implies that $\lim\limits_{n\to+\infty} \gamma_n$ exists and equal to $\frac{5\pi}{18}$.

13. Several families participated in a Christmas party. Each family consisted of mother, father, and at least one but not more than 10 children. Santa Claus chose a mother, a father, and one child from three different families for a sleigh ride. It turned out that he had exactly 3630 ways to choose such triples. How many children were in the party?

    Answer: 33

    Solution Let n be the number of the families and m is the total number of the children. First prove that the total number of ways to choose such triples is equal to $n(n-1)m$. Indeed, enumerate the families and assume that m_i is the number of children in the ith family. Then, if a child is chosen from the i family there is m_i ways to do it. Further in this case there is $n-1$ ways to choose a father from the rest of the families and after the choice of a father there is $n-2$ ways to choose a mother. So the total number of ways is equal to

    $$m_1(n-1)(n-2)+m_2(n-1)(n-2)+\ldots m_n(n-1)(n-2) = (m_1+m_2+\ldots+m_n)(n-1)(n-2) = m(n-1)(n-2).$$

    So, $(n-1)(n-2)m = 3630$. This shows that both $n-1$ and $n-2$ are divisors of 3630. Besides, since the number of children in each family is not greater that 10, then $m \leq 10n$, therefore $3630 < 10n^3$, which implies that $n^3 > 363$ and therefore $n \geq 8$.

    It remains to find all $n \geq 8$ such that $n-1$ and $n-2$ are divisors of 3630 . For this note that $3630 = 2 \cdot 3 \cdot 5 \cdot 11^2$ and from this it follows that the only n satisfying these properties is $n = 12$. Therefore the number of children $m = \frac{3630}{10 \cdot 11} = 33$.

14. Four ants simultaneously stand on the four vertices of a regular tetrahedron, with each ant at a different vertex. Simultaneously and independently, each ant moves from its vertex to one of the three adjacent vertices, each with equal probability. What is the probability that no two ants arrive at the same vertex?

    Answer $\frac{1}{9}$.

    Solution Since each ant can move from its vertex to any of three adjacent vertices, there are $3^4 = 81$ combinations of moves. Enumerate vertices somehow and try to understand for what combinations no two ants arrive at the same vertex. The ant from the first vertex can move to any of the rest of the vertex. Assume that he moves to vertex 2. Then the ant from the vertex 2 either moves to vertex 1 or to vertices 3 or 4. Consider these cases separately:

- If the ant from the vertex 2 moves to the vertex 1 then ants of the vertices 3 and 4 must swap (because they cannot move toward vertices 1 and 2). So, in this way we get a combination consisting of two swaps of ants in pairs of vertices $(1, 2)$ and $(3, 4)$.
- If the ant from the vertex 2 moves to the vertex 3, then the ant of the vertex 3 can move only to vertex 4. Indeed, he cannot move to vertex 2, because the ant from vertex 1 arrives there and he cannot move to vertex 1, because then in this case the ant of vertex 4 cannot move anywhere (all three vertices are already targets for other ants). Once the ant from the vertex 3 moves to the vertex 4 then the ant from vertex 4 must move to vertex 1 (as the only vertex that was not yet chosen as a target for other ants). In this way we get the following cyclic combination of moves: $1 \to 2 \to 3 \to 4 \to 1$. Exactly in the same way, if in the beginning of this item we assume that the vertex 2 moves to the vertex 4, then we must get the following cyclic combination of moves: $1 \to 2 \to 4 \to 3 \to 1$.

Combining two cases we get that if we chose that the ant from the first vertex moves to vertex 2, then we get 3 required combinations. Since there are 3 choices of the move of the ant from the first vertex, the total number of required combination of moves is equal to $3 \cdot 3 = 9$. Therefore the probability that no two ants arrive at the same vertex is equal to $\dfrac{9}{81} = \dfrac{1}{9}$.

15. Find the real number x such that there exists exactly one non-integer number among the four numbers $a = x - \sqrt{2}$, $b = x - \dfrac{1}{x}$, $c = x + \dfrac{1}{x}$, and $d = x^2 + 2\sqrt{2}$.

    Answer $x = \sqrt{2} - 1$.

    Solution First, either b or c is not integer. Indeed, assuming that both b and c are integer, we get that $x = \frac{b+c}{2}$ is rational, then both a and d are irrational, which contradicts the assumptions of the problem. Therefore both a and d must be integer. Note that

$$d = (x - \sqrt{2})^2 + 2\sqrt{2}(x - \sqrt{2}) + 2 + 2\sqrt{2} = a^2 + 2\sqrt{2}a + 2 + 2\sqrt{2} = a^2 + 2 + 2\sqrt{2}(a + 1).$$

    Since a and d are integer, the number $2\sqrt{2}(a + 1)$ must be integer, which is possible if and only if $a = -1$. Therefore, $x = \sqrt{2} - 1$. It remains, to check that among the numbers a, b, c, and d with this x there exists exactly one non-integer number. For this note that $\dfrac{1}{\sqrt{2} - 1} = \sqrt{2} + 1$. Then $a = -1$ is integer, $b = \sqrt{2} - 1 - (\sqrt{2} + 1) = -2$ is integer, $c = \sqrt{2} - 1 + \sqrt{2} + 1 = 2\sqrt{2}$ is not integer, $d = (\sqrt{2} - 1)^2 + 2\sqrt{2} = 2 - \cancel{2\sqrt{2}} + 1 + \cancel{2\sqrt{2}} = 3$ is integer, so indeed the only non-integer number is c.

16. Assume that $P_1(x), P_2(x), \ldots, P_6(x)$ are polynomials such that the sum of coefficients of the polynomial $P_i(x)$ is equal to i for each integer i between 1 and 6. What is the sum of the coefficients of the polynomial $F(x) = P_1(x)P_2(x) \ldots P_6(x)$, i.e. the polynomial obtained by taking the product of all polynomials $P_1(x), P_2(x), \ldots, P_6(x)$?

    Answer $6! = 720$.

    Solution Given a polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots a_1 x + a_0$ we have that the sum of its coefficients satisfy

$$a_n + a_{n-1} + \ldots + a_0 = f(1).$$

Therefore the sum of the coefficients of the polynomial $F(x)$ is equal to $F(1)$ which in turn is equal to $P_1(1)P_2(1)\dots P_6(1)$. Since by the assumption $P_i(1) = i$ we get that the requited sum of the coefficient is equal to $1 \cdot 2 \cdot \dots \cdot 8 = 6! = 720$.

17. Which is bigger π^e or e^π?

    Answer e^π.

    Solution Given two positive real numbers x_1 and x_2, $x_1^{x_2} > x_2^{x_1}$ if and only if $x_2 \ln x_1 > x_1 \ln x_2$ or, equivalently, $\dfrac{\ln x_1}{x_1} > \dfrac{\ln x_2}{x_2}$. Therefore if we define $f(x) = \dfrac{\ln x}{x}$, then the question is reduced to the comparison of $f(\pi)$ with $f(e)$. Note that $f'(x) = \dfrac{1-\ln x}{x^2}$. Therefore, $f'(e) = 0$ and $f'(x) < 0$ for $x > e$. Since $\pi > e$. this implies that $f(e) > f(\pi)$. Therefore $e^\pi > \pi^e$.

18. There are 16 students in class. Every month the teacher divides the class into two groups of 8 students each. After n months every two students were in different groups during at least one month. What is the minimal possible n?

    Answer $n = 4$.

    Solution

    First, show that $n \geq 4$. Indeed, assume that for a given n the teacher can make the required division. Then specifying for each month what group is the first group and what group is the second one, we can assign to each student a binary string of length n having 0 in the ith place if this student was in the first group during the ith month and 1 in the ith place if this student was in the second group during the ith month. The condition that every two students were in different groups during at least one month is equivalent to the fact that the binary strings corresponding to every two students are different. Note that number of different binary string is equal to 2^n, Therefore, $2^n \geq 16$, i.e. $n \geq 4$

    Now let us show that $n = 4$ is possible. For this take all possible different 16 binary strings of length 4 and assign each such string to exactly one of 16 students. Such assignments defines the schedule of required divisions.

19. Evaluate the following expression:

$$2016 \int_0^\pi |\sin(2015x)|\,dx - 2015 \int_0^\pi |\sin(2016x)|\,dx.$$

Answer 2.

Solution Given a positive integer n one has

$$\int_0^\pi |\sin(nx))|\,dx = n \int_0^{\pi/n} \sin(nx)\,dx.$$

Further, after the substitution $u = nx$:

$$\int_0^{\pi/n} \sin(nx)\,dx = \frac{1}{n}\int_0^\pi \sin(u)\,du = \frac{2}{n}.$$

So,

$$\int_0^{\pi} |\sin(nx))|\,dx = n\int_0^{\pi/n} \sin(nx)\,dx = \cancel{n}\frac{2}{\cancel{n}} = 2.$$

Thus,

$$2016\int_0^{\pi} |\sin(2015x)|\,dx - 2015\int_0^{\pi} |\sin(2016x)\,dx| = 2016\cdot 2 - 2015\cdot 2 = 2.$$

20. It is known that the number $\cos\dfrac{\pi}{5} - \cos\dfrac{2\pi}{5}$ is a rational number. Find this number (in lowest terms).

Answer $\dfrac{1}{2}$.

Solution. Note that if we take a regular n-gone on a coordinate plane with the center at the origin, then the sum of the position vectors of all its vertices is equal to zero (because the rotation in $\frac{2\pi}{n}$ around the origin will not change this sum). Taking the x-component of this sum and assuming that one of the vertices have an angle α with x-axis we get that for any α

$$\sum_{k=0}^{n-1} \cos\left(\alpha + \frac{2\pi k}{5}\right) = 0.$$

Taking $\alpha = \dfrac{\pi}{5}$ and $n = 5$, we get

$$\cos\frac{\pi}{5} + \cos\frac{3\pi}{5} + \cos\pi + \cos\frac{7\pi}{5} + \cos\frac{9\pi}{5} = 0. \tag{5}$$

Note that $\cos\dfrac{9\pi}{5} = \cos\dfrac{\pi}{5}$, because $\dfrac{9\pi}{5} = 2\pi - \dfrac{\pi}{5}$, and $\cos\dfrac{3\pi}{5} = \cos\dfrac{7\pi}{5} = -\cos\dfrac{2\pi}{5}$, because $\dfrac{7\pi}{5} = 2\pi - \dfrac{3\pi}{5}$ and $\dfrac{3\pi}{5} = \pi - \dfrac{2\pi}{5}$. Also, $\cos\pi = -1$. Substituting all this to (5) we get

$$2\left(\cos\frac{\pi}{5} - \cos\frac{2\pi}{5}\right) - 1 = 0,$$

i.e. $\cos\frac{\pi}{5} - \cos\frac{2\pi}{5} = \dfrac{1}{2}$.