2017 Power Team
Texas A&M High School Mathematics Contest
October 2017

- **In all problems prove your answers.**
- **If you are asked to describe a point, a set of points, or a path geometrically, it means that you have to give an explicit way how to construct them and even if you got your answer analytically you have to interpret it geometrically.**

1. Two villages $A$ and $B$ are located on one side of a river with healing water. The river bank is a straight line. Alice lives in village $A$ and her friend Bob lives in village $B$. One day Alice felt sick and she called Bob by phone asking him to bring her some fresh healing water from the river. So, Bob needs to go from village $B$ to village $A$ visiting the river. He decided to choose the shortest possible path to do this. Describe this path geometrically.

2. (a) An ant sits at a point $M$ inside of a given acute angle in the plane. He needs to visit one side of the angle, then the other side of the angle, and then to return to $M$ (note that visiting the vertex of the angle is considered as visiting both sides) . Describe geometrically the shortest path to perform this task.

   (b) Solve the same problem if the angle is obtuse or right.

3. (a) An ant crawls inside an acute triangle $ABC$. His task is to move along a closed path, visiting all sides of the triangle (if he visits a vertex it is considered as he visited both sides adjacent to this vertex). He can start anywhere inside the triangle. Describe the set of all points in which he can start to perform this task along the shortest possible path. Describe this path geometrically.

   (b) Solve the same problem if the triangle is not acute.

   In problems 4-8 below an ant crawls inside a **convex quadrilateral** $ABCD$ and again his task is to move along a closed path, visiting all sides of the quadrilateral (if he visits a vertex it is considered as he visited both sides adjacent to this vertex). He can start anywhere inside the quadrilateral.

4. Assume that the quadrilateral $ABCD$ is a rectangle with sides of length $a$ and $b$. Describe geometrically the shortest paths among all paths which perform the task and express the length of the shortest path in terms of $a$ and $b$.

5. Assume that the quadrilateral $ABCD$ satisfies the following condition: among all paths that perform the task there is a path of shortest possible length which does not meet a vertex of the quadrilateral. Prove that the quadrilateral $ABCD$ can be inscribed in a circle.

6. Assume that the quadrilateral $ABCD$ satisfies the condition of the previous problem, i.e. among all paths that perform the task there exists a path of the shortest possible length which does not meet a vertex of the quadrilateral. Show that among all paths that perform the task there are infinitely many different shortest paths (note that here we do not distinguish paths that trace the same closed curve with different starting points and/or directions of motion). Describe how to construct these shortest paths.

7. Assume that the quadrilateral $ABCD$ can be inscribed into a circle and the vertices do not belong to any semicircle. For such a quadrilateral, prove that among all paths that perform the task there is a path of shortest possible length which does not meet a vertex of the quadrilateral.

8. Assume that the quadrilateral $ABCD$ cannot be inscribed into a circle. Describe the algorithm to find the shortest path among the paths performing the task (you can use the fact that the shortest path exists without justifying it).