Solutions 2017 AB Exam

Texas A&M High School Math Contest
October 21, 2017

1. Solve for $x : \sqrt{x-2} = 4 - x$.

   ANSWER: $x = 3$

   Solution: $\sqrt{x-2} = 4 - x$
   $x - 2 = 16 - 8x + x^2$
   $x^2 - 9x + 18 = 0$
   $(x-6)(x-3) = 0$
   $x = 6, 3$ but $x = 6$ does not "check"

2. How many seven digit positive integers are palindromes? (A palindrome reads the same forwards and backwards, for example 1234321 is a palindrome.)

   ANSWER: 9000

   Solution: A seven digit palindrome has the form $xyzwzyx$ where of course $x \neq 0$. There are 9 choices for x, and 10 choices for each of y, z, w. This gives $9(10)(10)(10) = 9000$ palindromes.

3. When the positive integer n is divided by 39 the remainder is 10. What is the remainder when $5n$ is divided by 39?

   ANSWER: 11

   Solution: $n = 39q + 10$
   $$\begin{aligned} 5n = 5(39q) + 50 &= 39(5q) + 39 + 11 \\ &= 39(5q+1) + 11 \end{aligned}$$

4. If $f(x) = 3x + 4$ find a function g such that $f(g(x)) = 4x - 1$.

   ANSWER: $g(x) = \frac{4}{3}x - \frac{5}{3}$

   Solution: $$\begin{aligned} f(g(x)) = &3g(x) + 4 = 4x - 1 \\ &3g(x) = 4x - 5 \\ &g(x) = \tfrac{4}{3}x - \tfrac{5}{3} \end{aligned}$$

5. Suppose f is a linear function with $f(0) = 6$ and $f(2) = 3$. Find $f(1)$.

ANSWER: $\frac{9}{2}$

Solution: $f(x) = mx + b$

$$\begin{aligned} 6 =& f(0) = b \\ & f(x) = mx + 6 \\ 3 =& f(2) = 2m + 6 \\ & 2m = -3 \\ & m = -\tfrac{3}{2} \end{aligned}$$

$$f(x) = -\tfrac{3}{2}x + 6$$

$$f(1) = -\tfrac{3}{2} + 6 = -\tfrac{3}{2} + \tfrac{12}{2} = \tfrac{9}{2}$$

6. Find all values of m such that the line $y = mx + 3$ intersects the curve $y = x^2 + 2x + 7$ at exactly one point.

ANSWER: $m = 6, -2$

Solution: $mx + 3 = x^2 + 2x + 7$

$$x^2 + (2 - m)x + 4 = 0$$

We want the values of m so that this equation has a repeated root. For this the discriminant must be 0.

$$\begin{aligned} &(2 - m)^2 - 4(4) = 0 \\ &m^2 - 4m + 4 - 16 = 0 \\ &m^2 - 4m - 12 = 0 \\ &(m - 6)(m + 2) = 0 \\ &m = 6 \text{ or } -2 \end{aligned}$$

7. Find the minimum possible value of $2x^2 + 2xy + 4y + 5y^2 - x$.

ANSWER: $-\frac{5}{4}$

Solution: $x^2 + x^2 + 2xy + y^2 + 4y^2 + 4y - x$

$$\begin{aligned} &x^2 - x + (x + y)^2 + 4(y^2 + y) \\ &(x^2 - x + \tfrac{1}{4}) + (x + y)^2 + 4(y^2 + y + \tfrac{1}{4}) - \tfrac{1}{4} - 1 \\ &(x - \tfrac{1}{2})^2 + (x + y)^2 + 4(y + \tfrac{1}{2})^2 - \tfrac{5}{4} \end{aligned}$$

Note that $x = \frac{1}{2}, y = -\frac{1}{2}$ makes all three squared terms 0, so $-\frac{5}{4}$ is the absolute minimum.

8. For which base $b > 10$ is 103_b divided by 4_b equal to 29_b?

ANSWER: $b = 11$

Solution: $103_b = (4_b)(29_b)$ means

$$\begin{aligned} &b^2 + 3 = 4(2b+9) \\ &b^2 - 8b - 33 = 0 \\ &(b-11)(b+3) = 0 \\ &b = 11 \text{ or } -3 \end{aligned}$$

$b = -3$ is not a solution, since $b > 10$.

9. Find all integer pairs (a, b) such that $ab + a - 3b = 5$.

ANSWER: $(5, 0)$, $(4, 1)$, $(1, -2)$, $(2, -3)$

Solution: $ab + a - 3b = 5$

$$\begin{aligned} &a(b+1) - 3(b+1) = 5 - 3 \\ &(a-3)(b+1) = 2 \end{aligned}$$

Cases: $a - 3 = 1 \quad b + 1 = 2 \quad (4, 1)$

$a - 3 = -1 \quad b + 1 = -2 \quad (2, -3)$

$a - 3 = 2 \quad b + 1 = 1 \quad (5, 0)$

$a - 3 = -2 \quad b + 1 = -1 \quad (1, -2)$

10. Find a monic polynomial of degree 4 with integer coefficients having $\sqrt{2} - \sqrt{5}$ as a root.

ANSWER: $x^4 - 14x^2 + 9$

Solution: Let the polynomial be $x^4 + bx^3 + cx^2 + dx + e$. Since $\alpha = \sqrt{2} - \sqrt{5}$ is a solution, $\alpha^4 + b\alpha^3 + c\alpha^2 + d\alpha + e = 0$. Also, we get $\alpha^2 = 7 - 2\sqrt{10}, \alpha^3 = 17\sqrt{2} - 11\sqrt{5}, \alpha^4 = 89 - 28\sqrt{10}$. Substituting into the equation and combining like radicals, we get the system $-28 - 2c = 0, e + 7c + 89 = 0, 17b + d = 0, -11b - d = 0$, so $c = -14, e = 9, b = d = 0$. Thus, $x^4 - 14x^2 + 9$ is the only such polynomial.

11. Consider the infinite sequence of ordered pairs of integers:
$(1, 2017)$, $(2, 2018)$, $(3, 2019)$, $(4, 2020)$, ...
How many ordered pairs (a, b) are in this sequence where a divides b?

ANSWER: 36

Solution: The nth term in the sequence is $(n, n + 2016)$. We need to count the number of times n divides $n + 2016$.

$$\frac{n + 2016}{n} = 1 + \frac{2016}{n} \quad \cdot \quad 2016 = 2^5 \cdot 3^2 \cdot 7$$

2016 has $6 \cdot 3 \cdot 2 = 36$ positive integer divisors.

12. If the sum of the first n terms of an infinite sequence is $n^2 + 2n$, what is the 2017th term of the sequence?

ANSWER: 4035

Solution: Let the sequence be

$a_1, a_2, a_3, ..., a_n, ...$

We are given $a_1 + a_2 + ... + a_n = n^2 + 2n$
$$a_1 + a_2 + ... + a_{n-1} = (n-1)^2 + 2(n-1)$$

So

$$\begin{aligned} a_n &= (n^2 + 2n) - ((n-1)^2 + 2(n-1)) \\ &= n^2 + 2n - (n^2 - 2n + 1 + 2n - 2) \\ = \qquad &= 2n + 1 \end{aligned}$$

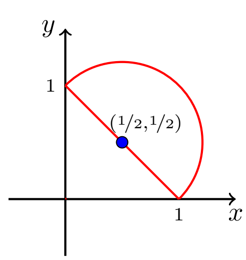
$$a_{2017} = 2(2017) + 1 = 4034 + 1 = 4035$$

13. Find the area of the region in the 1st quadrant bounded by the graphs of $y = 0$, $x = 0$, and $x^2 - x + y^2 - y = 0$.

ANSWER: $\frac{1}{2} + \frac{\pi}{4} = \frac{2+\pi}{4}$

Solution: $x^2 - x + y^2 - y = 0$
$$x^2 - x + \frac{1}{4} + y^2 - y + \frac{1}{4} = \frac{1}{2}$$
$$(x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 = (\frac{1}{\sqrt{2}})^2,$$

This is the circle with center $(1/2, 1/2)$ and radius $1/\sqrt{2}$ which contains the points $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$. The line segment from $(1,0)$ to $(0,1)$ is a diameter of the circle and the 'upperhalf' of the circle is in the 1st quadrant with area $\pi/4$. The remaining part of the circle in the 1st quadrant has area $1/2$.

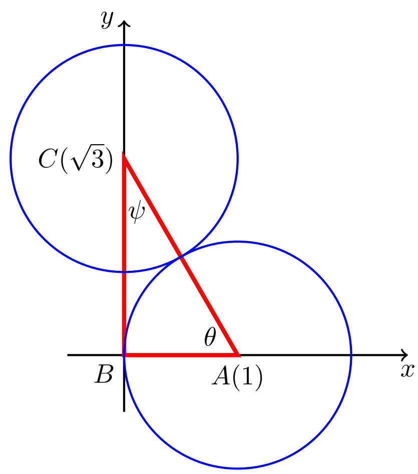


$$\text{Total area: } = \frac{1}{2} + \frac{\pi}{4} = \frac{2+\pi}{4}$$

14. A 1, 2, $\sqrt{3}$ right triangle has vertices A, B, C with B being the vertex at the right angle. A point P is randomly chosen within the triangle. What is the probability that P is within a distance of 1 from either A or C?

ANSWER: $\frac{\pi}{2\sqrt{3}} = \frac{\sqrt{3}}{6}\pi$

Solution:



The set of points within the triangle that are within distance 1 of the point A lie within a sector of a unit circle with angle $\theta = \pi/3$, and its area is $(\theta/2)r^2 = \pi/6$. Similarly those points within the triangle within a distance of 1 from C is a sector with angle $\psi = \pi/6$, and area $(\psi/2)r^2 = \pi/12$. The two regions above are disjoint except for the point $(\frac{1}{2}, \frac{\sqrt{3}}{2})$. So the area consisting of those points in the triangle within a distance of 1 from either A or C is

$$\frac{\pi}{6} + \frac{\pi}{12} = \frac{3\pi}{12} = \frac{\pi}{4}\,.$$

The area of the triangle is $\frac{1}{2}(1)(\sqrt{3}) = \frac{\sqrt{3}}{2}$ The probability of P being within 1 of A or C is

$$\frac{\pi/4}{\sqrt{3}/2} = (\frac{\pi}{4})(\frac{2}{\sqrt{3}}) = \frac{\pi}{2\sqrt{3}} = \frac{\sqrt{3}}{6}\pi$$

15. If the roots of $x^3 + ax^2 + bx + c$ are three consecutive positive integers, find the value of $\frac{a^2}{b+1}$.

ANSWER: 3

Solution: Let the roots be k, $k+1$, $k+2$. Then

$$\begin{aligned} a &= -(k+k+1+k+2) = -(3k+3) = -3(k+1) \text{ and } a^2 = 9(k+1)^2 \\ b+1 &= k(k+1) + (k+1)(k+2) + k(k+2) + 1 = 3k^2 + 6k + 3 \\ &= 3(k^2+2k+1) = 3(k+1)^2. \text{ Thus,} \\ \frac{a^2}{b+1} &= \frac{9(k+1)^2}{3(k+1)^2} = 3 \end{aligned}$$

16. Let A, B, C be randomly chosen and not necessarily different integers between 0 and 4 inclusive. What is the probability that $A + BC$ gives the same value as $(A + B) \cdot C$?

ANSWER: $\frac{9}{25}$

Solution: If A=AC, then either A=0, and C= 0, 1, 2, 3, or 4, or C=1, and A =0, 1, 2, 3, 4. There are 25 triples (0, B, C), 25 triples (A, B, 1), and 5 triples (0,B,1) are counted in both previous triples, so the total is 25+25-5=45. There are $5^3 = 125$ triples in total. The probability that $A + AC$ equals $(A + B) \cdot C$ is

$$\frac{45}{125} = \frac{9}{25}$$

17. Mike the chemist has a 200gm mixture that contains 60% metal A and 40% metal B. How many grams of metal A must he add to the mixture to have one that contains 80% metal A?

ANSWER: 200

Solution: Let x be the number of grams of metal A added to the original mixture. The new mixture now contains $200 + x$ grams and we want $.80(200 + x)$ to be metal A.

$$\begin{aligned} .80(200 + x) &= .60(200) + x \\ 160 + .8x &= 120 + x \\ 40 &= .2x \\ x &= \frac{40}{.2} = 200 \end{aligned}$$

18. Five different items in a store have lost their price tags. The five price tags are mixed up! Daisy the store clerk randomly attaches the tags to the five items. What is the probability that she attaches exactly two tags to correct items?

ANSWER: $\frac{1}{6}$

Solution: The two items having correct tags can be chosen in $\binom{5}{2} = 10$ different ways. There are now three tags left that must be placed incorrectly to each of the three remaining items. This can be done in two different ways.

The total number of ways to assign 5 tags to 5 items is $5! = 120$.

The probability of exactly two correct assignments is

$$\frac{10 \cdot 2}{120} = \frac{10}{60} = \frac{1}{6}$$

19. Find all polynomials P of degree 2 having real roots such that

$$P(x^2) = P(x)P(-x).$$

ANSWER: $x^2, \quad x(x-1) = x^2 - x, \quad (x-1)^2 = x^2 - 2x + 1$

Solution: Let $P(x) = \alpha x^2 + \beta x + \delta,$ then

$$P(-x) = \alpha x^2 - \beta x + \delta \text{ and}$$
$$P(x^2) = \alpha x^4 + \beta x^2 + \delta.$$
$$P(x^2) = P(x)P(-x) \text{ implies } \alpha^2 = \alpha.$$

Since $\alpha \neq 0$ then $\alpha = 1$ and $P(x)$ is monic.
If $x = a$ is a root of P then $P(a) = 0$. $P(a^2) = P(a)P(-a) = 0$ and a^2 is a root of P. So if a is a root of P then so is a^2. Hence if a is a root then a, a^2, a^4 are roots. But P has two real roots, so

$a^4 = a^2$ or $a^4 = a$
$a^4 - a^2 = 0$
$a^2(a^2 - 1) = 0 \Rightarrow a = 0$ or 1
$a^4 - a = 0$
$a(a^3 - 1) = a(a-1)(a^2 + a + 1) = 0 \Rightarrow a = 0, 1$
since $a^2 + a + 1$ has no real roots.

The possibilities for the two real roots of P
are $0, 0$ giving $P(x) = x^2$
$1, 0$ giving $P(x) = x(x-1)$
$1, 1$ giving $P(x) = (x-1)^2$. Each of these "works".

20. Find all ordered pairs of numbers (x, y) that satisfy both

$$x(x+y) = 9 \text{ and } y(y+x) = 16.$$

ANSWER: $(\frac{9}{5}, \frac{16}{5})$ and $(-\frac{9}{5}, -\frac{16}{5})$
Note that neither x nor y can equal zero.

Solution: $x + y = \dfrac{9}{x}$ and $x + y = \dfrac{16}{y}$

$$\frac{9}{x} = \frac{16}{y}$$
$$x = \frac{9}{16}y$$

From the first equation $x^2 + xy = 9$

$$\left(\frac{9}{16}y\right)^2 + \frac{9}{16}y^2 = 9$$
$$9 \cdot 25y^2 = 9 \cdot 16^2$$
$$y = \pm\frac{16}{5}$$

Using $x = \frac{9}{16}y$ gives $x = \pm\frac{9}{5}$.

21. How many positive integers less than or equal to 2016 are <u>not</u> relatively prime to 2016? (Note: $2016 = 2^5 \cdot 3^2 \cdot 7$)

ANSWER: 1440

Solution: An integer is <u>not</u> relatively prime to 2016 iff it is divisible by at least one of 2, 3, or 7. Also it is not relatively prime to 2016 iff it is not relatively prime to $2 \cdot 3 \cdot 7 = 42$.
We count the number of integers $1 \leq n \leq 42$ not relatively prime to 42.

Let A be the evens between 1 and 42 inclusive.

Let B be those divisible by 3.

Let C be those divisible by 7.

$$\begin{aligned}|A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\ &= \frac{42}{2} + \frac{42}{3} + \frac{42}{7} - \frac{42}{6} - \frac{42}{14} - \frac{42}{21} + 1 \\ &= 21 + 14 + 6 - 7 - 3 - 2 + 1 = 30.\end{aligned}$$

Every interval of 42 contains 30 integers not relatively prime to 42 and so not relatively prime to 2016. There are $\frac{2016}{42} = 48$ such intervals. hence $(30)(48) = 1440$ integers n, $1 \leq n \leq 2016$, not relatively prime to 2016.