Math Contest, Fall 2017
BC EXAM

1. List x, y, z in order from smallest to largest fraction:
$$x = \frac{111110}{111111},\ y = \frac{222221}{222223},\ z = \frac{333331}{333334}$$
Solution. Consider $1 - x = \frac{1}{111111}$, $1 - y = \frac{2}{222223} = \frac{1}{111111.5}$, $1 - z = \frac{3}{333334} = \frac{1}{11111.33333...}$, thus $1 - x > 1 - z > 1 - y$, and so $x < z < y$.
Answer. $x < z < y$, $\frac{111110}{111111} < \frac{333331}{333334} < \frac{222221}{222223}$.

2. A vase in a flower shop contains ten identical red and four identical pink roses. If you would like to pick one red and two pink ones, in how many ways can you do this?
Solution. There are ten choices for a red rose and $\frac{4 \cdot 3}{2}$ choices to pick two pink roses, thus there are $10 \cdot \frac{4 \cdot 3}{2} = 60$ ways to choose one red and two pink roses.
Answer. 60

3. Three-digit number $AB8$ is 296 larger than two-digit number AB. What is the value of the two-digit number AB?
Solution. $100A + 10B + 8 = 10A + B + 296$, $9(10A + B) = 288$, $10A + B = 32$, $AB = 32$
Answer. 32

4. The perimeter of an equilateral triangle is equal to the perimeter of a regular hexagon. What is the ratio of their areas (triangle area to hexagon area)?
Solution. If perimeters are equal, then the side a of the triangle is twice the side b of the hexagon. Then the triangle can be divided into four congruent equilateral triangles with side $\frac{a}{2}$, while hexagon - into six triangles of the same size. Thus, the ratio of areas is $4 : 6$ or $2 : 3$.
Answer. $4 : 6 = 2 : 3$.

5. When a number is divided by 5, the remainder is 2. What is the remainder when the number is multiplied by 8 and then divided by 5?
Solution. Denote the number by n. Then $n = 5q + 2$. Thus $8n = 40q + 16$ and so $8n = (8q + 3) \times 5 + 1$, the remainder is 1.
Answer. 1.

6. Jane has 6 sons and no daughters. Some of her sons have 6 sons, and the rest have none. Jane has a total of 30 sons and grandsons, and no great-grandsons. How many of Jane's sons and grandsons have no sons?
Solution. Since Jane has 6 sons, she has $30 - 6 = 24$ grandsons, of which none have sons. Of Jane's sons, $\frac{24}{6} = 4$ have sons, so $6 - 4 = 2$ do not have sons. Therefore, of Jane's sons and grandsons, $24 + 2 = 26$ do not have sons.
Answer. 26

7. Find all natural numbers n for which $2^n + 1$ is divisible by 3.
Solution. $2^n - (-1)^n = (2 + 1)(integer) = 3A$, so $2^n + 1 = (2^n - (-1)^n) + (-1)^n + 1 = 3A + (-1)^n + 1$. This number is divisible by 3 if and only if n is odd.
Answer. Any odd positive number $(n = 2k + 1, k = 0, 1, 2, ...)$

8. If John walks to his job from home and takes a bus on the way back, he spends 1.5 hour on round trip. If he takes a bus both ways, the round trip takes 30 min. How long (in hours) will it take him to walk both ways?
   Solution. Let w and b be respectively times to reach his work by walking and by taking a bus. It is given that $w + b = 1.5$ hr and $2b = .5$ hr. So $2w + 2b = 3$, $2w = 3 - .5 = 2.5$ hr
   Answer. 2.5 hr

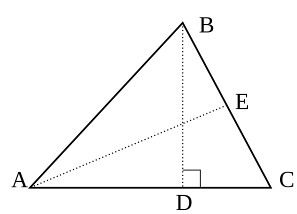
9. Find all possible values of an integer N such that $N^2 - 71$ is divisible by $7N + 55$.
   Solution. If $\frac{N^2-71}{7N+55} = K$ where K is an integer, then $N^2 - 7KN - (55K + 71) = 0$. Solving for N, we get $N = \frac{7K\pm\sqrt{49K^2+220K+284}}{2}$. The number under the radical must be a perfect square to get an integer N. Since $(7K+15)^2 < 49K^2+220K+284 < (7K+17)^2$, we conclude that $49K^2 + 220K + 284 = (7K + 16)^2$. Then, solving $49K^2 + 220K + 284 = (7K + 16)^2$, we get $K = 7$, and so $N = 57$ or $N = -8$.
   Answer. 57, -8

10. In an arithmetic sequence there are 10 numbers. The sum of terms at even places is 50, and the sum of terms at odd places is 35. Find the first term, a_1, and the difference, d, of this sequence.
    Solution. $a_1+a_3+a_5+a_7+a_9 = 35, a_2+a_4+a_6+a_8+a_{10} = 50$ or $5a_1+20d = 35, 5a_1+25d = 50$, from where $d = 3, a_1 = -5$.
    Answer. $a_1 = -5$, $d = 3$.

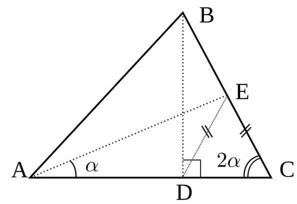
11. In a math class of 50 students, the average score on the final exam is 68. The best ten exams are all 100. Find the average of the other 40 exams.
    Solution. Denote the unknown average by A. Then $(40A + 10 \times 100)/50 = 68$ and thus $A = 60$.
    Answer. 60

12. If both roots of the quadratic equation $x^2 - 85x + c = 0$ are prime numbers, what is the value of c?
    Solution. Let us assume that two prime numbers, x_1 and x_2, are the solutions of the equation. Then $x_1 + x_2 = 85$. Since 85 is odd, either x_1 or x_2 must be even. The only even prime number is 2. Therefore, one number must be 2 and the other 83. Thus $c = 2 \times 83 = 166$.
    Answer. 166

13. In an acute triangle ABC, BD is the altitude and AE is the median. Given that the measure of the angle $\angle ECA$ is twice the measure of the angle $\angle EAC$ and that $BC = 10$ cm, find AD.



    Solution.



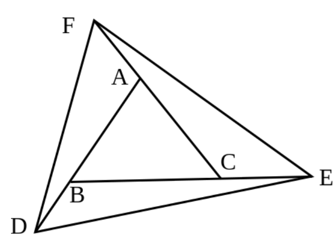
Denote $\angle EAC = \alpha$, then $\angle ECA = 2\alpha$. Draw the segment DE. It is a median to the hypotenuse in the right triangle DBC, and thus $DE = EC = 5$ cm, so triangle DEC is isosceles. Therefore, $\angle EDC = \angle ECD = 2\alpha$. Since $\angle ADE = 180° - 2\alpha$, we conclude that $\angle AED = \alpha$, thus triangle AED is isosceles, and $AD = DE = 5$ cm.
Answer. 5 cm

14. It is known that for some x the value of the expression $((x+2x)\cdot 3x - 4x) \div 5x$ stays the same even when all parenthesis are removed. What is the value of x?
Solution. It is given that $((x+2x)\cdot 3x - 4x) \div 5x = x + 2x \cdot 3x - 4x \div 5x$. Simplifying, we get $\frac{9x-4}{5} = 6x^2 + x - \frac{4}{5}$ or $6x^2 - \frac{4}{5}x = 0$. Since $x \neq 0$, we conclude that $6x - \frac{4}{5} = 0$ or $x = \frac{2}{15}$.
Answer. $\frac{2}{15}$

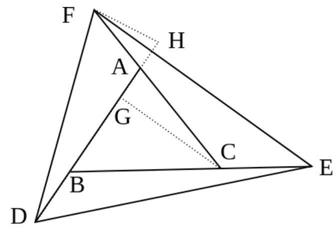
15. How many positive integers n have the property that when $1,000,063$ is divided by n, the remainder is 63?
Solution. Suppose n is a positive integer such that the remainder when $1,000,063$ is divided by n is 63. Then, $1,000,063 = nq + 63$ where $n > 63$. Thus, $1,000,000 = 10^6 = nq$. Now we need to count the divisors of 10^6 that are greater than 63. Notice that $10^6 = 2^6 \times 5^6$ has 49 positive divisors (the divisors are 2^k and 5^k for $k = 0, ...6$). Exactly twelve are less or equal to 63, namely 1, 2, 4, 8, 16, 32, 5, 10, 20, 40, 25, 50. Thus, the answer is $49 - 12 = 37$.
Answer. 37

16. Determine all integer values of the parameter a for which the equation $\frac{3}{a-2x} = \frac{5}{ax-4}$ has a negative solution.
Solution. $3ax - 12 = 5a - 10x$, $x = \frac{5a+12}{3a+10}$ which gives an interval for a: $-\frac{10}{3} < a < -\frac{12}{5}$. There is only one integer value there, -3.
Answer. -3

17. Each side of a triangle $\triangle ABC$ is extended as shown so that $BD = \frac{1}{2}AB, CE = \frac{1}{2}BC$, and $AF = \frac{1}{2}CA$. What is the ratio of the area of the triangle $\triangle DEF$ to the area of the triangle $\triangle ABC$?

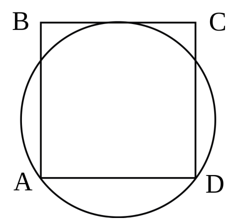


Solution.

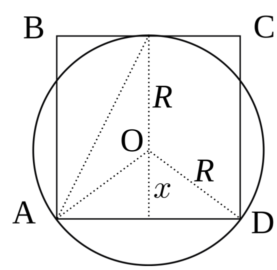


Let G be the point on AB such that CG is an altitude of $\triangle ABC$. Let H be the point on AD such that FH is an altitude of $\triangle ADF$. Then $\triangle AGC$ is similar to $\triangle AHF$ so $FH/CG = AF/AC = 1/2$. Next, $Area(\triangle ADF) = \frac{1}{2}(AD) \cdot (FH) = \frac{1}{2} \cdot \frac{3}{2} AB \cdot \frac{1}{2}(CG) = \frac{3}{4}(\frac{1}{2} \cdot AB \cdot CG) = \frac{3}{4} Area(\triangle ABC)$.
Similarly, $Area(\triangle BED) = (3/4) Area(\triangle ABC)$ and $Area(\triangle CFE) = (3/4) Area(\triangle ABC)$. Thus, $Area(\triangle DEF) = Area(\triangle ABC) + Area(\triangle ADF) + Area(\triangle BED) + Area(\triangle CFE) = \frac{13}{4} Area(\triangle ABC)$. Therefore, the ratio is $\frac{13}{4}$.
Answer. $\frac{13}{4}$

18. Given a square $ABCD$ with side length of 8 inches, a circle is drawn through vertices A and D and tangent to side BC. What is the radius of the circle in inches?

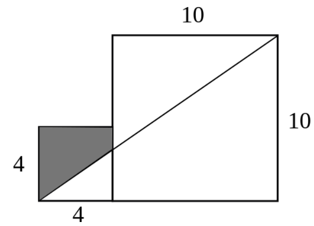


Solution.

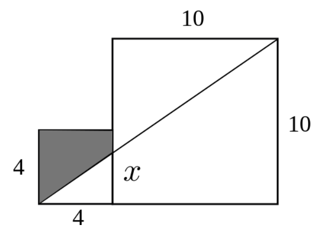


Denote the radius of the circle by R, the center of the circle by O, and the distance from the center to the side AD by x. Then $R + x = 8$ and from the right triangle with the hypotenuse AO, we get $R^2 - x^2 = 16$. Solving the system $R + x = 8$, $R^2 - x^2 = 16$, we get $R = 5$ (inches).
Answer. 5 inches

19. Consider two adjacent squares with sides 4 and 10 meters. Find the area of the shaded region.



Solution.



$\frac{x}{10} = \frac{4}{14}$, $x = \frac{20}{7}$. Thus the area is Area$=4^2 - \frac{4x}{2} = 16 - \frac{40}{7} = \frac{72}{7} = 10\,\frac{2}{7}$
Answer. $\frac{72}{7} = 10\,\frac{2}{7}$ $\ \mathrm{m}^2$

20. A circle is inscribed into a right triangle. From the center of the circle a perpendicular is dropped on the hypotenuse, dividing the hypotenuse into two segments of length 12 and 5. Find the legs of the triangle.
Solution. Here the legs of the triangle are equal to $a = 12 + r$, $b = 5 + r$ where r is the radius of the circle. Then, notice that $a = 7 + b$ and $a^2 + b^2 = 17^2$, so we get the equation for b: $(7 + b)^2 + b^2 = 289$, $b^2 + 7b - 120 = 0, b = 8$. Then $a = 8 + 7 = 15$.
Answer. $15, 8$ (or $8, 15$)

21. Consider a square $ABCD$ and choose points E and F on the square such that $\triangle BEF$ is equilateral. What is the ratio of the area of $\triangle DEF$ to that of $\triangle ABE$?
Solution. Assume, without loss of generality, that the side length of $ABCD$ is 1. Denote $DE = x$, then $AE = 1 - x$. Since triangle $\triangle BEF$ is equilateral, right triangles $\triangle EAB$ and $\triangle FCB$ are congruent, so $CF = AE$ and $DE = DF$. Thus, the ratio of areas is $\frac{x^2}{2} \div \frac{(1-x)(1)}{2} = \frac{x^2}{1-x}$. To get rid of x, let us notice that $BE = EF = x\sqrt{2}$. Then, applying the Pythagorean Theorem to $\triangle ABE$, we get $(1 - x)^2 + 1 = 2x^2$ or $x^2 + 2x = 2$, so $x^2 = 2 - 2x = 2(1 - x)$. Thus, the ratio of areas is $\frac{x^2}{1-x} = 2$.
Answer. 2.

22. Given a sequence of integers $\{a_n\}$ such that $a_1 = 1$ and $a_{m+n} = a_m + a_n + mn$ for $m, n = 1, 2, ...,$ find a_{15}.
Solution. Take $n = 1$. Then $a_{m+1} = a_m + a_1 + m$, so $a_{m+1} - a_m = m + 1$. Thus $a_{15} - a_{14} = 15$, $a_{14} - a_{13} = 14$, ..., $a_2 - a_1 = 2$. Adding these equalities up, we get $a_{15} - a_1 = 15 + 14 + ... + 2$. So $a_{15} = 15 + 14 + ... + 2 + 1 = \frac{(15)(16)}{2} = 120$.
Answer. 120