CD Exam Solutions
Texas A&M High School Math Contest
October 21, 2017

All answers must be simplified, and if units are involved, be sure to include them.

1. Solve the equation $\sqrt{x+7} = x+1$.

   Solution: Squaring both sides of the equation we get

   $$x+7 = x^2+2x+1 \Leftrightarrow x^2+x-6=0 \Leftrightarrow x=2 \text{ or } x=-3.$$

   By replacing $x=2$ and $x=-3$ in our initial equation we see that only $x=2$ is a solution of our equation. For $x=-3$ we get $\sqrt{4}=-2$ which is false.

   Answer: 2

2. It can be shown that the number

   $$A = \sqrt{|2\sqrt{7}-8|} - \sqrt{8+2\sqrt{7}} + 2019$$

   is an integer. Find its value.

   Solution:

   $$\begin{aligned} A &= \sqrt{8-2\sqrt{7}} - \sqrt{8+2\sqrt{7}} + 2019 = \sqrt{(\sqrt{7}-1)^2} - \sqrt{(\sqrt{7}+1)^2} + 2019 \\ &= \sqrt{7}-1-(\sqrt{7}+1)+2019 = 2017. \end{aligned}$$

   Answer: 2017

3. The atomic number of antimony is one less than four times the atomic number of aluminium. If twice the atomic number of aluminium is added to that of antimony, the result is the atomic number of the element iridium, which is 77. What is the atomic number of antimony?

   Solution: Let x be the atomic number of antimony and y be the atomic number of aluminium. Then we have

   $$\begin{cases} x &= 4y-1 \\ 2y+x &= 77 \end{cases} \Leftrightarrow \begin{cases} x &= 4y-1 \\ x &= 77-2y \end{cases} \Leftrightarrow \begin{cases} x &= 4y-1 \\ 4y-1 &= 77-2y \end{cases} \Leftrightarrow \begin{cases} x &= 4y-1 \\ 6y &= 78. \end{cases} \Leftrightarrow \begin{cases} x &= 51 \\ y &= 13. \end{cases}$$

   The atomic number of antimony is 51.

   Answer: 51

4. Find the radius of the circle with the equation

   $$2x^2+2y^2+4x+24y+73=0.$$

   Solution: By completing the squares we get

   $$2x^2+4x+2+2y^2+24y+72-1=0 \Leftrightarrow 2(x+1)^2+2(y+6)^2=1 \Leftrightarrow (x+1)^2+(y+6)^2=\frac{1}{2}.$$

   So the radius of the circle is $\sqrt{\dfrac{1}{2}} = \dfrac{\sqrt{2}}{2}$.

   Answer: $\dfrac{1}{\sqrt{2}}$ or $\dfrac{\sqrt{2}}{2}$

5. If n is a positive integer then the expression

$$A = \frac{3^{n+1} \cdot 4^n + 3^n \cdot 4^{n+1} + 12^{n+1}}{2^{n+1} \cdot 3^{n+1} - 5 \cdot 6^n}$$

is also a positive integer. Find $\sqrt[n]{A/19}$.

Solution: We have that

$$A = \frac{3 \cdot 3^n \cdot 4^n + 3^n \cdot 4 \cdot 4^n + 12 \cdot 12^n}{2 \cdot 2^n \cdot 3 \cdot 3^n - 5 \cdot 6^n} = \frac{3 \cdot (3 \cdot 4)^n + 4 \cdot (3 \cdot 4)^n + 12 \cdot 12^n}{6 \cdot (2 \cdot 3)^n - 5 \cdot 6^n}$$
$$= \frac{3 \cdot 12^n + 4 \cdot 12^n + 12 \cdot 12^n}{6 \cdot 6^n - 5 \cdot 6^n} = \frac{12^n(3+4+12)}{6^n(6-5)} = \frac{12^n \cdot 19}{6^n} = 19 \cdot 2^n.$$

Answer: 2

6. To lay wall-to-wall carpeting in a rectangular shaped living room and dining room takes 612 ft^2 of carpet. The living room is 3 ft longer than it is wide. The dining room floor is 6 ft wider than the width of the living room, and its length is twice the width of the living room. Find the width of the living room.

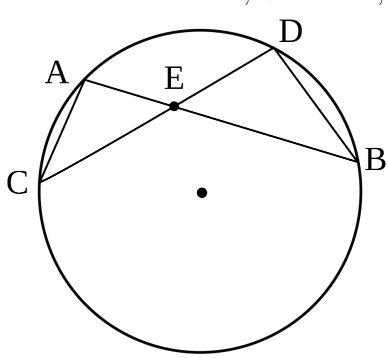
Solution: Let x be the width of the living room in feet. Then the length of the living room is $x+3$ feet which implies that the area of the living room is $x(x+3)$ ft^2. The width of the dining room is $x+6$ and the length of the dining room is $2x$. Therefore, the area of the dining room is $2x(x+6)$ ft^2. We get the equation

$$x(x+3) + 2x(x+6) = 612 \Leftrightarrow x^2 + 5x - 204 = 0 \Leftrightarrow x = -17 \text{ or } x = 12.$$

Since x is positive, we conclude that the width of the living room is 12 ft.

Answer: 12 ft

7. In the circle below, $CD = 14$, $AE = 3$, and $EB = 16$. Find $CE^2 + ED^2$.



Solution: We notice that $\triangle AEC$ and $\triangle DEB$ are similar. This implies

$$\frac{AE}{DE} = \frac{CE}{BE} \Leftrightarrow AE \cdot EB = CE \cdot ED.$$

After replacing the values from the text and using that $ED = DC - CE = 14 - CE$ we get that

$$48 = CE(14 - CE) \Leftrightarrow CE^2 - 14CE + 48 = 0 \Leftrightarrow (CE-6)(CE-8) = 0 \Leftrightarrow CE = 6 \text{ or } CE = 8.$$

If $CE = 6$, then $ED = 8$, and if $CE = 8$, then $ED = 6$. In both cases, $CE^2 + ED^2 = 36 + 64 = 100$.

Answer: 100

8. If $1072^x = 8$ and $67^y = 8$, find $\dfrac{1}{x} - \dfrac{1}{y}$.

Solution: From the two equations we get that

$$1072 = 8^{\frac{1}{x}} \text{ and } 67 = 8^{\frac{1}{y}}.$$

Dividing the two equations we obtain

$$\frac{1072}{67} = 8^{\frac{1}{x}-\frac{1}{y}} \Leftrightarrow 16 = 8^{\frac{1}{x}-\frac{1}{y}} \Leftrightarrow 2^4 = 2^{3\left(\frac{1}{x}-\frac{1}{y}\right)} \Leftrightarrow 4 = 3\left(\frac{1}{x}-\frac{1}{y}\right) \Leftrightarrow \frac{1}{x}-\frac{1}{y} = \frac{4}{3}.$$

Answer: $\dfrac{4}{3}$

9. Find the last digit of the number

$$A = 142^1 + 142^2 + 142^3 + \cdots + 142^{20}.$$

Solution: We can write A as

$$\begin{aligned} A &= (142^1 + 142^2 + 142^3 + 142^4) + (142^5 + 142^6 + 142^7 + 142^8) \\ &\quad + \cdots + (142^{17} + 142^{18} + 142^{19} + 142^{20}) \\ &= (142^1 + 142^2 + 142^3 + 142^4) + 142^4(142^1 + 142^2 + 142^3 + 142^4) \\ &\quad + \cdots + 142^{16}(142^1 + 142^2 + 142^3 + 142^4). \end{aligned}$$

The last digits of 142^1, 142^2, 142^3, and 142^4 are 2, 4, 8, and 6, respectively. Therefore, the last digit of $142^1 + 142^2 + 142^3 + 142^4$ is 0 since $2 + 4 + 8 + 6 = 20$. We conclude that the last digit of A is 0.

Answer: 0

10. Find the sum of the squares of all solutions of the equation $|x^2 - 2x - 25| = 10$.

Solution: Our equation is equivalent to

$$\begin{gathered} x^2 - 2x - 25 = 10 \text{ or } x^2 - 2x - 25 = -10 \\ x^2 - 2x - 35 = 0 \text{ or } x^2 - 2x - 15 = 0 \\ (x+5)(x-7) = 0 \text{ or } (x+3)(x-5) = 0 \\ x = -5 \text{ or } 7 \text{ or } x = -3 \text{ or } 5. \end{gathered}$$

The sum of the squares of all solutions is $(-5)^2 + 7^2 + (-3)^2 + 5^2 = 108$.

Answer: 108

11. A community center plans to have a fund raising concert. There are 200 VIP tickets and 1000 general tickets. A VIP ticket costs \$100 more than a general ticket. What should be the minimum price of a general ticket so that the funds raised would be at least \$80,000?

Solution: Let x be the price of a general ticket. Then the price of a VIP ticket is $x + 100$. If all tickets are sold then the funds raised would total

$$200(x + 100) + 1000x = 1200x + 20000.$$

So we need

$$1200x + 20000 \geq 80000 \Leftrightarrow 1200x \geq 60000 \Leftrightarrow x \geq 50.$$

Therefore, the minimum price of a general ticket should be \$50.

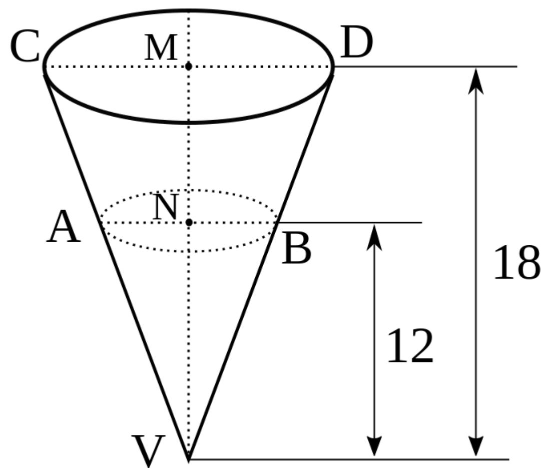
Answer: \$50

12. How many positive integers n have the property that when $1,000,063$ is divided by n, the remainder is 63?

Solution. Suppose n is a positive integer such that the remainder when $1,000,063$ is divided by n is 63. Then, $1,000,063 = nq + 63$ where $n > 63$. Thus, $1,000,000 = 10^6 = nq$. Now we need to count the divisors of 10^6 that are greater than 63. Notice that $10^6 = 2^6 \times 5^6$ has 49 positive divisors (the divisors are 2^k and 5^k for $k = 0, \ldots 6$). Exactly twelve are less or equal to 63, namely 1, 2, 4, 8, 16, 32, 5, 10, 20, 40, 25, 50. Thus, the answer is $49 - 12 = 37$.

Answer: 37

13. The diagram below shows a cup in the form of an inverted cone of height 18 cm and base radius 7.5 cm. It is filled with water to depth of 12 cm. Find the additional volume of water required to fill up the cup. Express your answer in terms of π.



Solution: In the diagram, $\triangle VNB$ is similar to $\triangle VMD$ which implies that

$$\frac{NB}{MD} = \frac{VN}{VM} \Leftrightarrow \frac{NB}{7.5} = \frac{12}{18} \Leftrightarrow NB = 5.$$

We found that the radius of the water surface is 5 cm. So the volume of water is equal to $\frac{1}{3}\pi \cdot NB^2 \cdot VN = 100\pi\ cm^3$. We also know that

$$\frac{\text{Volume of water}}{\text{Volume of the cup}} = \left(\frac{12}{18}\right)^3 \Leftrightarrow \frac{100\pi}{\text{Volume of the cup}} = \frac{8}{27}.$$

Therefore, the volume of the cup is $337.5\pi\ cm^3$ which gives us that the required volume of water to fill up the cup is $337.5\pi - 100\pi = 237.5\pi\ cm^3$.

Answer: $237.5\pi\ cm^3$

14. Find $3a + 5b$ if the polynomial $P(x) = 2ax^3 + 2bx^2 + 3x + 5$ is divisible by $x + 1$ and the sum of its coefficients is a positive and prime number.

Solution: From the fact that $P(x)$ is divisible by $x + 1$ we get that $P(-1) = 0$ which is equivalent to $b - a + 1 = 0$. The sum of the coefficients of $P(x)$ is $2a + 2b + 3 + 5 = 2(a + b + 4)$ which must be a prime number. Therefore, $a + b + 4 = 1$. The system with equations $b - a + 1 = 0$ and $a + b + 4 = 1$ has the solution $a = -1$ and $b = -2$. So $3a + 5b = -13$.

Answer: -13

15. Simplify

$$\frac{1^2}{1^4+1}+\frac{2^2-1}{2^4+2}+\frac{3^2-2}{3^4+3}+\cdots+\frac{1000^2-999}{1000^4+1000}.$$

Solution: We notice that all terms are of the form

$$\begin{aligned}\frac{n^2-(n-1)}{n^4+n}&=\frac{n^2-n+1}{n(n^3+1)}=\frac{n^2-n+1}{n(n+1)(n^2-n+1)}\\&=\frac{1}{n(n+1)}=\frac{(n+1)-n}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}.\end{aligned}$$

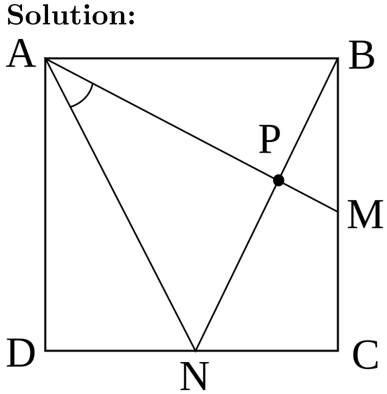
Therefore, our expression is equal to

$$\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{1000}-\frac{1}{1001}\right)=1-\frac{1}{1001}=\frac{1000}{1001}.$$

Answer: $\frac{1000}{1001}$

16. Let $ABCD$ be a square and M and N be the midpoints of the sides BC and CD, respectively. Find $\sin(\angle MAN)$.

Solution:



Let $AB=a$ and let P be the intersection point of AM and BN. We see that $\triangle BPM$ and $\triangle BCN$ are similar since $\angle AMB=\angle BNC$. We get that

$$\frac{BP}{BC}=\frac{BM}{BN}\Leftrightarrow BP=\frac{BC\cdot BM}{BN}.$$

Applying the Pythagorean Theorem in $\triangle BCN$ gives us that $BN=\frac{a\sqrt{5}}{2}$ and together with the above equation we get that $BP=\frac{a\sqrt{5}}{5}$. It implies that $PN=BN-BP=\frac{3\sqrt{5}a}{10}$. Since $\triangle APN$ is a right triangle ($\angle APN=\angle BPM=\angle BCN=90°$) and $AN=BN=\frac{a\sqrt{5}}{2}$, we get that

$$\sin(\angle MAN)=\frac{PN}{AN}=\frac{\frac{3\sqrt{5}a}{10}}{\frac{a\sqrt{5}}{2}}=\frac{3}{5}.$$

Answer: 0.6 or $\frac{3}{5}$

17. Consider the expression

$$E(x) = \frac{1}{x^3} + \frac{1}{x^2} + x^2 + x^3.$$

If a is a solution of the equation $x + \dfrac{1}{x} = 3$, find $E(a)$.

Solution: Since a is a solution of the equation $x + \dfrac{1}{x} = 3$ we get that

$$a + \frac{1}{a} = 3 \Rightarrow \left(a + \frac{1}{a}\right)^2 = 9 \Leftrightarrow a^2 + 2 + \frac{1}{a^2} = 9 \Leftrightarrow a^2 + \frac{1}{a^2} = 7.$$

Next we see that

$$\left(a^2 + \frac{1}{a^2}\right)\left(a + \frac{1}{a}\right) = 7 \cdot 3 = 21 \Leftrightarrow a^3 + a + \frac{1}{a} + \frac{1}{a^3} = 21 \Rightarrow a^3 + \frac{1}{a^3} = 21 - 3 = 18.$$

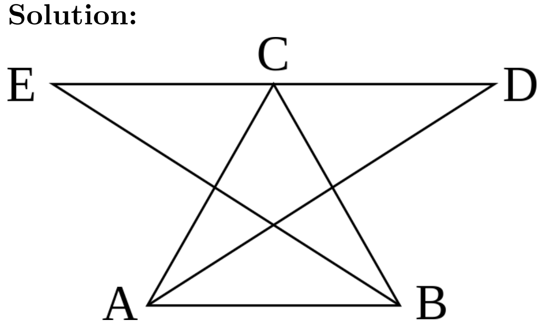
Therefore,

$$E(a) = \left(a^2 + \frac{1}{a^2}\right) + \left(a^3 + \frac{1}{a^3}\right) = 7 + 18 = 25.$$

Answer: 25

18. Let $\triangle ABC$ be an acute triangle and let D and E be such that BC is a perpendicular bisector of AD and AC is a perpendicular bisector of BE. Find $\angle ACB$ if the points D, C, and E are collinear.

Solution:



Since BC is a perpendicular bisector of AD we get that $\angle ACB = \angle BCD$, and since AC is a perpendicular bisector of BE we get that $\angle ACB = \angle ACE$. Therefore, $\angle ACB = \angle BCD = \angle ACE$ which implies that $\angle DCE = 3\angle ACB$. If the points D, C, and E are collinear, then $\angle DCE = 180°$, and we conclude that $\angle ACB = 60°$.

Answer: $60°$

19. Find the minimum value of the expression

$$E(x, y) = 5(x^2 + y^2) - 2(-x + 4xy + 8y) + 36,$$

where x and y are real numbers.

Solution: We write $E(x, y)$ as

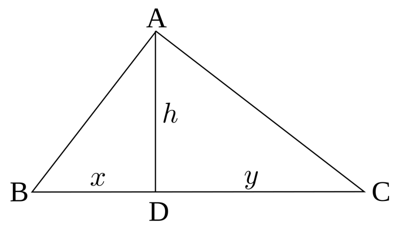
$$\begin{aligned} E(x, y) &= 5x^2 + 5y^2 + 2x - 8xy - 16y + 36 = 5x^2 + 2x(1 - 4y) + 5y^2 - 16y + 36 \\ &= 5x^2 + 2 \cdot \sqrt{5}x \cdot \frac{1 - 4y}{\sqrt{5}} + \left(\frac{1 - 4y}{\sqrt{5}}\right)^2 + 5y^2 - 16y + 36 - \left(\frac{1 - 4y}{\sqrt{5}}\right)^2 \\ &= \left(\sqrt{5}x + \frac{1 - 4y}{\sqrt{5}}\right)^2 + \frac{9y^2 - 72y + 179}{5} \\ &= \left(\sqrt{5}x + \frac{1 - 4y}{\sqrt{5}}\right)^2 + \frac{9}{5}(y - 4)^2 + 7. \end{aligned}$$

Therefore, the minimum value of $E(x, y)$ is 7 and it is attained when $y - 4 = 0$ and $\sqrt{5}x + \dfrac{1-4y}{\sqrt{5}} = 0$ ($y = 4$ and $x = 3$).

Answer: 7

20. In the right triangle ABC we have that $\angle A = 90°$ and D belongs to the side BC so that AD is perpendicular to BC. Find $\tan(\angle ABC)$ if the area of $\triangle ABC$ is $2AD^2$ and $\angle ABC > \angle ACB$.

Solution:



Let $BC = a$, $AD = h$, $BD = x$, and $DC = y$. The area of $\triangle ABC$ is equal to $\dfrac{AD \cdot BC}{2} = \dfrac{ah}{2}$ on one hand and $2AD^2 = 2h^2$ on the other hand. So, $\dfrac{ah}{2} = 2h^2$, which implies that $a = 4h$. Since $\triangle ABC$ is a right triangle with $\angle A = 90°$ and AD is perpendicular to BC, we get that $xy = h^2$. Also, $x + y = a = 4h$ implies that $y = 4h - x$. Therefore, $x(4h - x) = h^2 \Leftrightarrow x^2 - 4xh + h^2 = 0$. Using the quadratic formula we obtain that $x = (2+\sqrt{3})h$ or $x = (2-\sqrt{3})h$. If $x = (2+\sqrt{3})h$, then $y = (2-\sqrt{3})h$, and if $x = (2-\sqrt{3})h$, then $y = (2+\sqrt{3})h$. But $x < y$ because $\angle ABC > \angle ACB$. Therefore, $x = (2-\sqrt{3})h$ and

$$\tan(\angle ABC) = \tan(\angle ABD) = \frac{AD}{BD} = \frac{h}{x} = \frac{1}{2-\sqrt{3}} = 2+\sqrt{3}.$$

Answer: $\dfrac{1}{2-\sqrt{3}}$ or $2+\sqrt{3}$