EF Exam
Texas A&M High School Math Contest
October, 2017

1. A drawer contains 100 red socks, 80 green socks, 60 blue socks, and 40 black socks. What is the smallest number of socks that must be selected (without looking) to guarantee that the selection contains at least 10 pairs (a pair is two socks of the same color).

The answer is 23. If the number of socks of one color (say red) is even, then they all will be used as pairs; if it is odd, then only one sock will be "wasted". If the number of chosen socks is odd, then it is not possible that all colors will have an odd number of chosen socks, since then the sum of four odd numbers is even. It follows that if we choose 23 socks, then at most 3 socks will be without pair, and we will get at least 10 pairs. On the other hand, 22 socks is not enough, since, for example, it could be 19 red socks, and one of each of the remaining colors.

2. Alice rolls a fair regular octahedral die marked with the numbers 1 through 8. Then Bob rolls a fair six-sided die. What is the probability that the product of two rolls is a multiple of 3?

The product is a multiple of 3 if and only if Alice's or Bob's number is a multiple of 3. The probability that Alice gets a number divisible by 3 is $\frac{2}{8} = \frac{1}{4}$. The probability that Bob gets a number divisible by 3 is $\frac{2}{6} = \frac{1}{3}$. The probability that both of them will get numbers divisible by 3 is $\frac{1}{4} \cdot \frac{1}{3} = \frac{1}{12}$. It follows that the probability that the product is a multiple of 3 is $\frac{1}{4} + \frac{1}{3} - \frac{1}{12} = \frac{3+4-1}{12} = \frac{1}{2}$.

3. Find the minimum value of $\sqrt{x^2 + y^2}$ when $5x + 12y = 60$.

The minimum value is equal to the distance from the line $5x+12y = 60$ to $(0, 0)$, i.e., to the height to the hypotenuse of the right triangle with the vertices $(12, 0)$, $(0, 0)$, $(0, 5)$. The area of this triangle is $\frac{5\cdot 12}{2} = 30$. The length of the hypotenuse is $\sqrt{5^2 + 12^2} = 13$. It follows that the height is $\frac{2\cdot 30}{13} = \frac{60}{13}$.

4. Find the number of solutions in positive integers of $2x + 3y = 2017$.

We have $1 \le y \le 671$. For every value of y, the only possible value of x is $\frac{2017-3y}{2}$. This number is an integer if and only if the number $2017 - 3y$ is even, which happens if and only if y is odd. It follows that all the possible values of y are $1, 3, 5, \ldots, 671$, and there are $672/2 = 336$ possibilities.

5. An insect lives on the surface of a regular tetrahedron with edges of length 1. It wishes to travel on the surface of the tetrahedron from the midpoint of one edge to the midpoint of the opposite edge (the edge that has no common endpoint with the first edge). What is the length of the shortest such trip?

The insect has to travel through at least two faces, since the opposite edges do not belong to one face. If we unfold two adjacent faces on the plane, we get a parallelogram formed by two equilateral triangles. The midpoints of the edges will become midpoints of the opposite sides of the parallelogram. The shortest trip between the midpoints of the opposite sides of the parallelogram is along the straight line, and is equal to the side of the parallelogram, i.e., to 1.

6. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. Find

$$\sum_{n=1}^{1024} \lfloor \log_2 n \rfloor.$$

The sequence $\lfloor \log_2 n \rfloor$ is equal to k for the values $2^k \le n \le 2^{k+1} - 1$. The sum of $\lfloor \log_2 n \rfloor$ for these values of n is equal to $k(2^{k+1} - 2^k) = k2^k$. It follows that the sum in the problem question is equal to $1 \cdot 2 + 2 \cdot 2^2 + \cdots + 9 \cdot 2^9 + 10$. The sum $x + 2x^2 + 3x^3 + \cdots + nx^n$ can be found by differentiating the sum $f(x) = 1 + x + x^2 + \cdots + x^n$. Namely, $f(x) = \frac{x^{n+1}-1}{x-1}$ and $f'(x) = 1 + 2x + 3x^2 + \cdots nx^{n-1}$, so $x + 2x^2 + 3x^3 + \cdots nx^n = xf'(x) = x\frac{(n+1)x^n(x-1)-x^{n+1}+1}{(x-1)^2} = \frac{x(nx^{n+1}-(n+1)x^n+1)}{(x-1)^2}$. It follows that $1 + 2 \cdot 2^2 + \cdots + 9 \cdot 2^9 = 2(9 \cdot 2^{10} - 10 \cdot 2^9 + 1) = 2(9 \cdot 1024 - 5120 + 1) = 2(9217 - 5120) = 2 \cdot 4097 = 8194$. The answer to the problem is $8194 + 10 = 8204$.

7. Find all x in the interal $[0, \pi]$ such that $\sin^8 x + \cos^8 x = \frac{41}{128}$.

We have $\cos 2x = 2\cos^2 x - 1$, so $\cos^2 x = (\cos 2x + 1)/2$. Similarly, $\sin^2 x = (1 - \cos 2x)/2$. It follows that $\cos^8 x = (\cos 2x + 1)^4/16$ and $\sin^8 x = (1 - \cos 2x)^4/16$. We have to solve

$$\frac{(1-\cos 2x)^4 + (\cos 2x + 1)^4}{16} = \frac{41}{128},$$

i.e.,

$$(1-\cos 2x)^4 + (\cos 2x + 1)^4 = \frac{41}{8}.$$

Open the parenthesis, using the binomial formula: $(1-\cos 2x)^4 + (\cos 2x + 1)^4 = 1 - 4\cos 2x + 6\cos^2 2x - 4\cos^3 2x + \cos^4 2x + \cos^4 2x + 4\cos^3 2x + 6\cos^2 2x + 4\cos 2x + 1 = 2 + 12\cos^2 2x + 2\cos^4 2x$, so that our equation is equivalent to

$$\cos^4 2x + 6\cos^2 2x + 1 = \frac{41}{16}.$$

Consider it as a quadratic equation $y^2 + 6y - \frac{25}{16} = 0$ with respect to $y = \cos^2 2x$. Its solutions are

$$y = \frac{-6 \pm \sqrt{36 + 4 \cdot \frac{25}{16}}}{2} = \frac{-6 \pm \frac{13}{2}}{2} = \frac{-12 \pm 13}{4}.$$

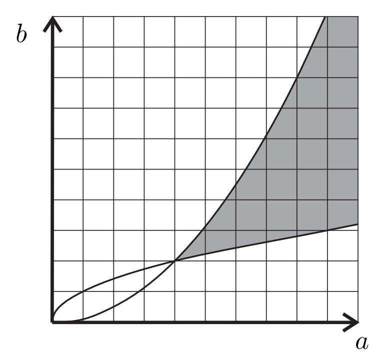
Note that $y = \cos^2 2x$ is non-negative, so $y = \frac{1}{4}$, and $\cos 2x = \pm\frac{1}{2}$. It follows that possible values of $2x$ in the interval $[0, 2\pi]$ are $\pi/3, 2\pi/3, 4\pi/3, 5\pi/3$, hence the values of x in the interval $[0, \pi]$ are $\pi/6, \pi/3, 2\pi/3, 5\pi/6$.

8. If a and b are positive real numbers such that each of the equations $x^2 + ax + 2b = 0$ and $x^2 + 2bx + a = 0$ has real roots, then what is the smallest possible value of $a + b$?

Computing the discriminants of the equations, we get $a^2 - 8b \geq 0$ and $4b^2 - 4a \geq 0$. It follows that the set of points (a, b) of the plain satisfying the conditions of the problem is the intersection of the regions below the parabolas $b = a^2/8$ and $a = b^2$ shown on the figure. The smallest value of $a + b$ is attained at the intersection of the parabolas, which is found by solving the system

$$\begin{cases} b = a^2/8 \\ a = b^2 \end{cases}$$

We get $b = b^4/8$, i.e., $b^3 = 8$, so $b = 2$ and $a = 4$. Consequently, the smallest value of $a + b$ is 6.



9. For which of the numbers $n = 2017$ or $n = 2016$ is the polynomial $x^{2n} + (x+1)^{2n} + 1$ divisible by $x^2 + x + 1$?

Since $x^{2n} + (x+1)^{2n} + 1 = x^{2n} + ((x^2 + x + 1) - x^2)^{2n} + 1$, the polynomial $x^{2n} + (x+1)^{2n} + 1$ is divisible by $x^2 + x + 1$ if and only if $x^{2n} + (-x^2)^{2n} + 1 = x^{4n} + x^{2n} + 1$ is divisible by $x^2 + x + 1$.

Since $x^3 - 1$ is divisible by $x^2 + x + 1$, every polynomial $x^m - x^{m-3}$ is divisible by $x^2 + x + 1$, and $x^m - x^{m-3k} = (x^m - x^{m-3}) + (x^{m-3} - x^{m-6}) + \cdots + (x^{m-3k+3} - x^{m-3k})$ is divisible by $x^2 + x + 1$.

Suppose that $n = 3k$. Then $x^{4n} - 1 + x^{2n} - 1$ is divisible by $x^2 + x + 1$, so $x^{4n} + x^{2n} + 1$ gives residue 3 when divided by $x^2 + x + 1$.

If $n = 3k + 1$, then $2n = 6k + 2$, $4n = 12k + 4 = 3(4k + 1) + 1$. Consequently, $x^{4n} - x + x^{2n} - x^2$ is divisible by $x^2 + x + 1$, hence $x^{4n} + x^{2n} + 1 = x^{4n} - x + x^{2n} - x^2 + x^2 + x + 1$ is also divisble by $x^2 + x + 1$.

If $n = 3k+2$, then $2n = 3(2k+1)+1$, $4n = 3(4k+2)+2$, and $x^{4n}+x^{2n}+1 = x^{4n}-x^2+x^{2n}-x+x^2+x+1$ is divisible by x^2+x+1.

Consequently, $x^{2n}+(x+1)^{2n}+1$ is divisible by x^2+x+1 if and only if n is *not* divisible by 3, so that the answer is $n = 2017$.

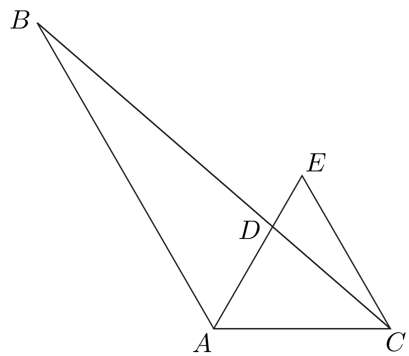
10. Find all pairs of integers (x, y) such that $x^3 + y^3 = 91$.

Consider the case when both x and y are non-negative. Then they are less than 5, and x^3 and y^3 belong to the set $\{0, 1, 8, 27, 64\}$. The only two numbers in this set whose sum is 91 are 27 and 64.

Suppose that x is positive and y is negative. Denote $z = -y$. Then we have $x^3 - z^3 = 91$. The difference between two consecutive cubes of natural numbers is $(n+1)^3 - n^3 = 3n^2 + 3n + 1$, and it is bigger than 91 for $n > 5$. It follows that $z \in \{1, 2, 3, 4, 5\}$. Then $z^3 \in \{1, 8, 27, 64, 125\}$ and $z^3 + 91 \in \{92, 99, 118, 155, 216\}$. The only cube of an integer in the latter set is $216 = 6^3$. We get the only case $x = 6, z = 5$. It follows that the list of all pairs (x, y) are $(3, 4), (4, 3), (6, -5), (-5, 6)$.

11. Point D is on side CB of triangle ABC. If $\angle CAD = \angle DAB = 60°$, $AC = 1$, and $AB = 2$, find AD.

Draw the line parallel to $\overline{AB}$ through C, and let E be its intersection with $\overline{AD}$. Then $\triangle ACE$ is equilateral, so that $EC = AE = AC = 1$. The triangles $\triangle EDC$ and $\triangle ADB$ are similar, hence $AD : DE = 2 : 1$, which implies that $AD = \frac{2}{3}$.



12. Evaluate the integral $\int_{-1}^{15} \frac{dx}{\sqrt{x+10}-\sqrt{x+1}}$.

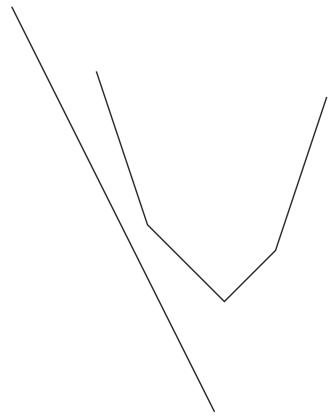
We have $\int_{-1}^{15} \frac{dx}{\sqrt{x+10}-\sqrt{x+1}} = \int_{-1}^{15} \frac{\sqrt{x+10}+\sqrt{x+1}}{x+10-x-1}\, dx = \frac{1}{9}\int_{-1}^{15} \sqrt{x+10}+\sqrt{x+1}\, dx$. We have $\int_{-1}^{15} \sqrt{x+10}\, dx = \frac{2}{3}(x+10)^{3/2}\Big|_{x=-1}^{15} = \frac{2}{3}(125-27) = \frac{196}{3}$, and $\int_{-1}^{15} \sqrt{x+1}\, dx = \frac{2}{3}(x+1)^{3/2}\Big|_{x=-1}^{15} = \frac{128}{3}$. It follows that our integral is equal to $\frac{1}{9}\left(\frac{196}{3} + \frac{128}{3}\right) = 12$.

13. Positive integers a, b, c are such that $a < b < c$, and the system

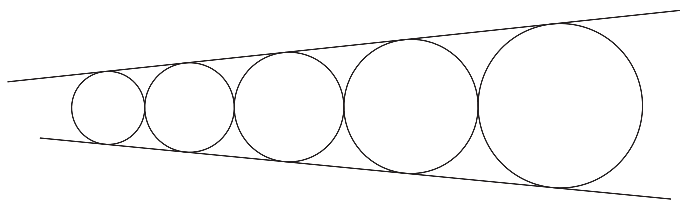
$$\left\{ \begin{array}{rcl} 2x + y & = & 2017 \\ |x-a| + |x-b| + |x-c| & = & y \end{array} \right.$$

has exactly one solution. What is the smallest possible value of c?

The graph of the function $y = |x-a| + |x-b| + |x-c|$ consists of the half-line $y = (a+b+c) - 3x$ of slope -3 from $-\infty$ to $x = a$, the segment of the line $y = (-a+b+c) - x$ of slope -1 on the interval from $x = a$ to $x = b$, the segment of the line $y = (-a-b+c) + x$ of slope 1 on the interval from $x = b$ to $x = c$, and the half-line $y = (-a-b-c) + 3x$ of slope 3. The other equation is a line of slope 2. We see that the graph of the first function is convex, and that the graphs have a unique intersection point if and only if the solution satisfies $x = a$. Then $y = b + c - 2a$, and the numbers a, b, c have to satisfy $2a + b + c - 2a = 2017$, i.e., $b + c = 2017$. We have $1 \leq a < b < c$, hence $2017 = b + c < 2c$, so the smallest value of c is not less than $2018/2 = 1009$. The numbers $a = 1, b = 1008, c = 1009$ show that the smallest value of c is 1009.



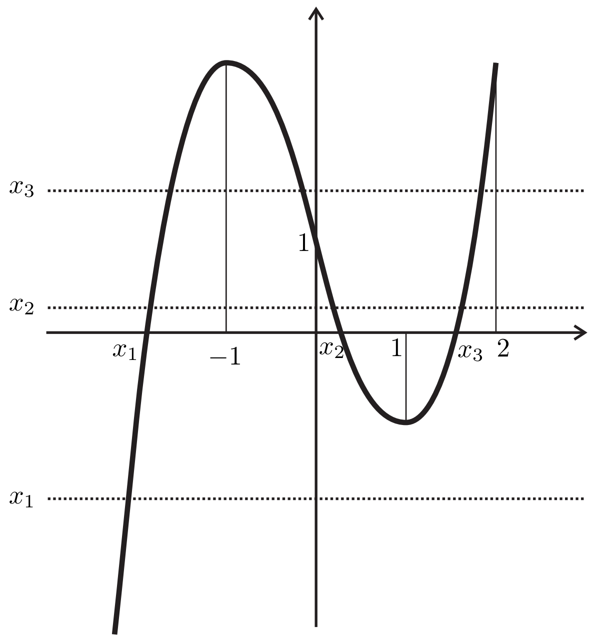
14. Five circles are tangent to one another consecutively and to the lines L_1 and L_2. Find the radius of the middle circle if the radius of the largest circle is 18 and the radius of the smallest one is 8.



Every circle is uniquely determined by the lines and the previous circle. There exists a similarity transformation (a homothety) with center in the intersection point of the lines L_1 and L_2 transforming the smallest circle to the next circle. By the above mentioned uniqueness, this homothety will map the second circle to the third, etc.. It follows that the ratio of the radii of two tangent circles is always the same, so that they form a geometric progression. It follows that the radii are of the form $8, 8q, 8q^2, 8q^3, 8q^4 = 18$, hence $q^4 = 18/8 = 9/4$, so that $q^2 = 3/2$, and the middle circle is of radius $\frac{8 \cdot 3}{2} = 12$.

15. Let $f(x) = x^3 - 3x + 1$. Find the number of distinct real roots of the polynomial $f(f(x))$.

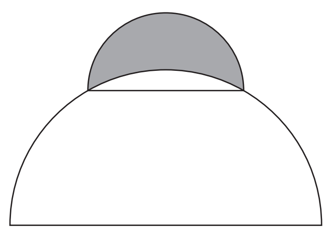
Taking derivative $f'(x) = 3x^2 - 3$, we get that the critical points of $f(x)$ are $x = \pm 1$. We have $f(-1) = 3$, $f(1) = -11$. The function $f(x)$ increases on the interval $(-\infty, -1)$, then decreases on the interval $(-1, 1)$, and increases on the interval $(1, \infty)$, see the figure below. Note that $f(2) = 8 - 6 + 1 = 3 > 0$. It follows that there are three roots of $x^3 - 3x + 1$: one in each of the intervals $(-\infty, -1), (-1, 1)$, and $(1, 2)$. Let us denote them $x_1 < x_2 < x_3$. Then the number of distinct real roots of $f(f(x))$ is equal to the sum of the numbers of solutions of the equations $f(x) = x_i$. The equation $f(x) = x_1$ has one real solution, $f(x) = x_2$ has three real solutions, $f(x) = x_3$ has three real solutions. It follows that $f(f(x))$ has seven real roots.



16. Let A_1 and C_1 be points on the sides BC and AB, respectively, of the triangle $\triangle ABC$ such that AA_1 and CC_1 are perpendicular to $\overline{BC}$ and $\overline{AB}$, respectively. If $AA_1 \geq BC$ and $CC_1 \geq AB$, what are possible measures of the angles of the triangle $\triangle ABC$?

Since a leg of a right triangle is less than the hypotenuse, we have $AA_1 \leq AB$ and $CC_1 \leq BC$. We get $BC \leq AA_1 \leq AB$ and $AB \leq CC_1 \leq BC$. It follows that $AB = BC = AA_1 = CC_1$, hence $\triangle ABC$ is an isosceles right triangle, so its angles are $\angle BAC = \angle BCA = \pi/4$ and $\angle ABC = \pi/2$.

17. A semicircle of diameter 1 sits at the top of a semicircle of diameter 2, as shown. What is the area of the shaded area inside the smaller semicircle and outside the larger semicircle?



The area of the common part of the semicircle is equal to the area of the sector of angle $\pi/3$ of the bigger circle minus the area of the equilateral triangle of side 1, i.e., to $\pi/6 - \sqrt{3}/4$. It follows that the area of the area inside the smaller semicircle and outside the larger semicircle is $\pi/8 - (\pi/6 - \sqrt{3}/4) = \sqrt{3}/4 - \pi/24$.

18. Let $f(x)$ be a function defined on the non-negative real numbers and taking non-negative real values, such that:

(i) $f(xf(y))f(y) = f(x + y)$ for all $x, y \geq 0$;

(ii) $f(2) = 0$;

(iii) $f(x) \neq 0$ for all $0 \leq x < 2$.

Find $f(\sqrt{2}) + f(\pi)$.

Take $x = 2 - y$ and $0 \leq y < 2$ in (i):

$$0 = f(2) = f((2 - y)f(y))f(y).$$

Since $f(y) \neq 0$, we have $f((2-y)f(y)) = 0$, hence, by (iii), $(2-y)f(y) \geq 2$.

Take $x = \frac{2}{f(y))}$ in (i):

$$f(x+y) = f(xf(y))f(y) = f(2)f(y) = 0,$$

hence $x + y \geq 2$, i.e., $\frac{2}{f(y)} \geq 2 - y$ for every $0 \leq y < 2$. This inequality is equivalent to $2 \geq (2-y)f(y)$.

We have proved that for all $0 \leq y < 2$ we have $(2-y)f(y) \geq 2$ and $(2-y)f(y) \leq 2$. In other words, we have showed that $(2-y)f(y) = 2$, i.e., $f(y) = \frac{2}{2-y}$ for all $0 \leq y < 2$.

Suppose now $z \geq 2$. Then, taking $x = z - 2$ and $y = 2$ in (i), we get

$$f(z) = f(z - 2 + 2) = f((z-2)f(2))f(2) = 0.$$

It follows that the only possible function is

$$f(x) = \begin{cases} \frac{2}{2-x} & \text{if } x \in [0, 2), \\ 0 & \text{if } x \geq 2, \end{cases}$$

and the answer is $\frac{2}{2-\sqrt{2}} = 2 + \sqrt{2}$.

19. Evaluate the product

$$(\sqrt{3} + \tan 1^\circ)(\sqrt{3} + \tan 2^\circ) \cdots (\sqrt{3} + \tan 29^\circ).$$

Let P be the product. Then $\frac{P}{2^{29}} = \frac{\frac{\sqrt{3}}{2}\cos 1^\circ + \frac{1}{2}\sin 1^\circ}{\cos 1^\circ} \cdot \frac{\frac{\sqrt{3}}{2}\cos 2^\circ + \frac{1}{2}\sin 2^\circ}{\cos 2^\circ} \cdots \frac{\frac{\sqrt{3}}{2}\cos 29^\circ + \frac{1}{2}\sin 29^\circ}{\cos 29^\circ} = \frac{\cos(30^\circ - 1^\circ)}{\cos 1^\circ} \cdot \frac{\cos(30^\circ - 2^\circ)}{\cos 2^\circ} \cdots \frac{\cos(30^\circ - 29^\circ)}{\cos 29^\circ} = 1$. Consequently, the product is equal to 2^{29}.

20. Let $E(n)$ be the largest integer k such that 5^k divides $2^2 3^3 4^4 \cdots n^n$. Find $\lim_{n\to\infty} \frac{E(n)}{n^2}$.

There are $\lfloor \frac{n}{5^k} \rfloor$ numbers divisible by 5^k among the numbers $2, 3, 4, \ldots, n$. It follows that $E(n)$ is equal to $(5 + 10 + \cdots + 5\lfloor \frac{n}{5} \rfloor) + (25 + 50 + \cdots + 25\lfloor \frac{n}{25} \rfloor) + \cdots + (5^{\lfloor \log_5 n \rfloor} + 2 \cdot 5^{\lfloor \log_5 n \rfloor} + \cdots + \lfloor \frac{n}{5^{\lfloor \log_5 n \rfloor}} \rfloor \cdot 5^{\lfloor \log_5 n \rfloor})$, i.e., to

$$5 \cdot \frac{\lfloor \frac{n}{5} \rfloor \left(\lfloor \frac{n}{5} \rfloor + 1 \right)}{2} + 25 \cdot \frac{\lfloor \frac{n}{25} \rfloor \left(\lfloor \frac{n}{25} \rfloor + 1 \right)}{2} + \cdots + 5^k \frac{\lfloor \frac{n}{5^k} \rfloor \left(\lfloor \frac{n}{5^k} \rfloor + 1 \right)}{2},$$

where $k = \lfloor \log_5 n \rfloor$. It follows, that for some numbers $r_i \in [0, 1)$ we have

$$E(n) = 5 \cdot \frac{\left(\frac{n}{5} + r_1\right)\left(\frac{n}{5} + 1 + r_1\right)}{2} + 25 \cdot \frac{\left(\frac{n}{25} + r_2\right)\left(\frac{n}{25} + 1 + r_2\right)}{2} + \cdots + 5^k \cdot \frac{\left(\frac{n}{5^k} + r_k\right)\left(\frac{n}{5^k} + 1 + r_k\right)}{2},$$

hence

$$\frac{E(n)}{n^2} = 5 \cdot \frac{\left(\frac{1}{5} + \frac{r_1}{n}\right)\left(\frac{1}{5} + \frac{1+r_1}{n}\right)}{2} + 25 \cdot \frac{\left(\frac{1}{25} + \frac{r_2}{n}\right)\left(\frac{1}{25} + \frac{1+r_2}{n}\right)}{2} + \cdots + 5^k \cdot \frac{\left(\frac{1}{5^k} + \frac{r_k}{n}\right)\left(\frac{1}{5^k} + \frac{1+r_k}{n}\right)}{2}.$$

This expression is not more than

$$L_n = 5 \cdot \frac{1}{2 \cdot 5^2} + 5^2 \cdot \frac{1}{2 \cdot 5^4} + \cdots + 5^k \cdot \frac{1}{2 \cdot 5^{2k}} = \frac{1}{2}\left(\frac{1}{5} + \frac{1}{5^2} + \cdots + \frac{1}{5^k}\right) \to \frac{1}{10(1 - 1/5)} = \frac{1}{8}$$

and less than

$$U_n = 5 \cdot \frac{\left(\frac{1}{5} + \frac{1}{n}\right)\left(\frac{1}{5} + \frac{2}{n}\right)}{2} + 25 \cdot \frac{\left(\frac{1}{25} + \frac{1}{n}\right)\left(\frac{1}{25} + \frac{2}{n}\right)}{2} + \cdots + 5^k \cdot \frac{\left(\frac{1}{5^k} + \frac{1}{n}\right)\left(\frac{1}{5^k} + \frac{2}{n}\right)}{2}.$$

We have

$$5^m \cdot \frac{\left(\frac{1}{5^m} + \frac{1}{n}\right)\left(\frac{1}{5^m} + \frac{2}{n}\right)}{2} - 5^m \cdot \frac{1}{2 \cdot 5^{2m}} = \frac{5^m}{2}\left(\frac{3}{5^m n} + \frac{2}{n^2}\right) = \frac{3}{2n} + \frac{5^m}{n^2}.$$

It follows that

$$U_n - L_n = \frac{3k}{2n} + \frac{5 + 25 + \cdots + 5^k}{n^2} = \frac{3\lfloor \log_5 n \rfloor}{2n} + \frac{5 \cdot \frac{5^{\lfloor \log_5 n \rfloor + 1} - 1}{4}}{n^2} \leq$$

$$\frac{3 \log_5 n}{2n} + \frac{5^{\log_5 n + 2} - 5}{4n^2} = \frac{3 \log_5 n}{2n} + \frac{25n - 5}{4n^2},$$

which converges to 0 as $n \to \infty$. Consequently, $\lim_{n\to\infty} U_n = \lim_{n\to\infty} L_n = \frac{1}{8}$ is the limit of $\frac{E(n)}{n^2}$.

21. Evaluate, for every positive integer n, the integral

$$\int \frac{x^n}{1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}}\, dx.$$

$$\int \frac{x^n}{1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}}\, dx = n! \int \frac{\left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}\right) - \left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!}\right)}{1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}}\, dx =$$

$$n! \int \left(1 - \frac{1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!}}{1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}}\right) dx$$

Since $1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!}$ is equal to the derivative of $1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$, the integral is equal to

$$n! \left(x - \ln\left|1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}\right|\right) + C.$$