BC EXAM

Texas A&M High School Math Contest

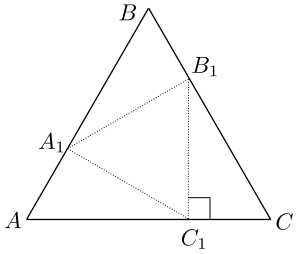
October 20 2018

Directions: All answers must be simplified, and if units are involved, include them in your answer.

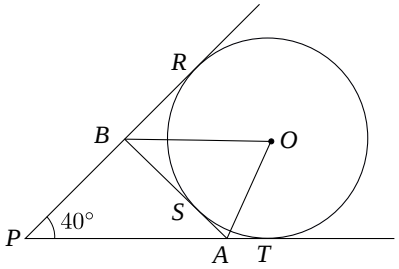
1. Two distinct polynomials $x^2 + ax + b$ and $x^2 + bx + a$ share a linear factor. Find $a + b$.

2. The figure below suggests how to stack n^2 equal circles and wrap a wire around them. What is the length of the shortest wire that wraps around a stack of 2025 circles of radius 1?

3. In an equilateral triangle $\triangle ABC$, segments AA_1, BB_1 and CC_1 are equal segments. If $\angle B_1C_1C$ is a right angle, find the ratio of the area of $\triangle A_1B_1C_1$ to the area of $\triangle ABC$.



4. The triangle PAB is formed by three tangents to a circle centered at the point O and $\angle APB = 40°$. Find $\angle AOB$.



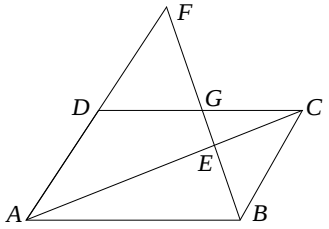
5. A function f satisfies the following conditions for all positive integers n.

$$f(2n) = f(n), \qquad f(2n+1) = f(n) + 1, \qquad f(1) = 1$$

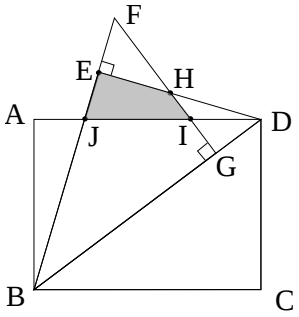
Find the smallest n such that $f(n) = 7$.

6. From a two digit number N we subtract the number with the digits reversed and find that the result is a positive cube. Find all possible N.

7. A point F is taken on the extension of side AD of a parallelogram $ABCD$ as shown below. The segment BF intersects diagonal AC at E and the side DC at G. If $EF = 32$ and $GF = 24$, find BE.



8. Consider a rectangle $ABCD$ with $AB = 3$ and $BC = 4$. Reflect a right triangle $\triangle BCD$ along the diagonal BD to obtain a right triangle $\triangle BDE$, and then rotate $\triangle BCD$ about the vertex B to obtain a right triangle $\triangle BGF$. Let points H, I and J be the intersections between segments as below. Find the area of a quadrilateral $EJIH$.



9. If $x = \sqrt{3 - \sqrt{8}}$, find $x^7 + \dfrac{1}{x^7}$.

10. Find the number of all possible solutions of the equation $xyz = 8000$ when x, y and z are positive integers.

11. Let n be a three digit positive integer. Define a function $f(n)$ by

    $f(n) =$ (the sum of the digits of n) + (the sum of the products of two digits of n) + (the product of the digits of n).

    For example, if $n = 234$,

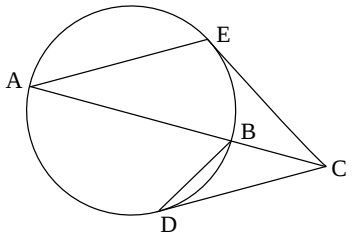
    $$f(n) = (2 + 3 + 4) + (2 \cdot 3 + 3 \cdot 4 + 4 \cdot 2) + (2 \cdot 3 \cdot 4).$$

    Find all possible three digit positive integers n such that $f(n) = n$.

12. Three radars are spaced 6, 8, and 10 miles from each other on the ground, which is assumed to be horizontal. The radars spot an airplane at a distance of 13 miles at the same time. What is the elevation of the airplane?

13. Find all integer solutions (x, y) of the equation $15x^2 - 5xy - 16x + 7y + 6 = 0$.

14. Let AB be a diameter of a circle. A point C is chosen on the extension of AB beyond B. Points D and E are chosen on the circle so that $BC = BD$ and $EA = EC$. Find the ratio $BC : EC$ if CD is tangent to the circle.



15. Given a natural number n, four students A, B, C, and D claimed as follows.

    A: $20 < n < 50$.

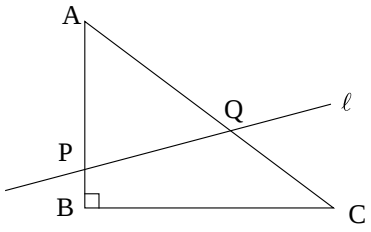
    B: n is a divisor of 120.

    C: n has 8 divisors (natural numbers)

    D: n is a multiple of 12.

    If one and only one student made a false statement, who is it?

16. A line ℓ bisects both of the perimeter and the area of a right triangle $\triangle ABC$ as in the picture below. Find AQ if $AB = 3$, $BC = 4$.



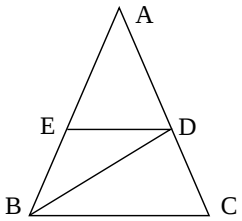
17. Find all pairs (x, y) satisfying the system $\begin{cases} 2x^2 + 7xy + 6y^2 = 12 \\ 7x^2 + 20xy + 14y^2 = 23. \end{cases}$

18. Let A and B be two positive integers and let

$$\begin{aligned} A + B &= C \\ B + C &= D \\ C + D &= E \\ &\vdots \\ L + M &= N \\ &\vdots \\ X + Y &= Z. \end{aligned}$$

Find G if $A + B + C + \cdots + J = 990$.

19. An isosceles $\triangle ABC$ is made out of 3 smaller isosceles $\triangle AED$, $\triangle EBD$, and $\triangle BCD$ with $AE = AD$, $ED = EB$, $BD = BC$, and $AB = AC$. Find the area of $\triangle BCD$ if the area of $\triangle ABC$ is 1.



20. A triangle has sides $x^2 + x + 1$, $2x + 1$ and $x^2 - 1$. Find the largest interior angle of the triangle.