BC EXAM SOLUTIONS

Texas A&M High School Math Contest
October 20 2018

Directions: All answers must be simplified, and if units are involved, include them in your answer.

1. Two distinct polynomials $x^2 + ax + b$ and $x^2 + bx + a$ share a linear factor. Find $a + b$.

   Answer. $a + b = -1$

   Solution. Let $x - \gamma$ be the common factor. Then $x = \gamma$ is a common root of those two polynomials. This yields

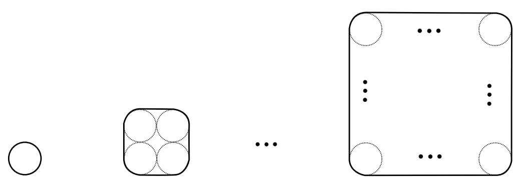
   $$\gamma^2 + a\gamma + b = 0 \quad \text{and} \quad \gamma^2 + b\gamma + a = 0. \tag{1}$$

   Subtracting these, we get

   $$(a - b)\gamma + b - a = 0 \Rightarrow (a - b)(\gamma - 1) = 0 \tag{2}$$

   Since the polynomials are distinct, we get $a \neq b$ and thus (2) implies $\gamma = 1$. Plugging this into (1) we have $a + b = -1$.

2. The figure below suggests how to stack n^2 equal circles and wrap a wire around them. What is the length of the shortest wire that wraps around a stack of 2025 circles of radius 1?

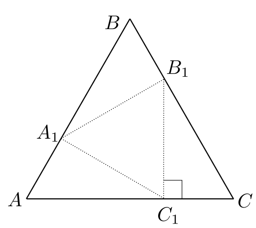


   Answer. $352 + 2\pi$

   Solution. Since $2025 = 45^2$, we have a square of 45 circles by 45 circles. The four round corners form a circle, which gives us 2π for the length of wire. The distance between the two outer circles on each side is $2 \cdot (45 - 1)$. Thus the length of the wire is

   $$4 \cdot 2 \cdot (45 - 1) + 2\pi = 352 + 2\pi.$$

3. In an equilateral triangle $\triangle ABC$, segments AA_1, BB_1 and CC_1 are equal segments. If $\angle B_1C_1C$ is a right angle, find the ratio of the area of $\triangle A_1B_1C_1$ to the area of $\triangle ABC$.



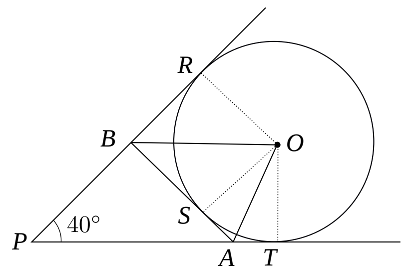
   Answer. $\dfrac{1}{3}$

   Solution. The triangle $\triangle CC_1B_1$ is a right triangle with $\angle C = 60°$. Thus if $C_1C = a$ then $CB_1 = 2a$ and $B_1C_1 = \sqrt{3}a$ (You may think of the right half of an equilateral triangle with side length 2). Since it can be shown that all three smaller triangles involving (i.e., containing as the vertices) the points A_1, B_1 and C_1 are congruent we have

$$\frac{B_1C_1}{AC} = \frac{B_1C_1}{AC_1 + C_1C} = \frac{\sqrt{3}a}{2a + a} = \frac{\sqrt{3}}{3}.$$

So the ratio of the area of $\triangle A_1B_1C_1$ to the area of $\triangle ABC$ is $\left(\frac{\sqrt{3}}{3}\right)^2 = \frac{1}{3}$.

4. The triangle PAB is formed by three tangents to a circle centered at the point O and $\angle APB = 40°$. Find $\angle AOB$.



Answer. $\angle AOB = 70°$

Solution. The sum of all angles of the quadrilateral $PTOR$ is $360°$. Since $\angle T$ and $\angle R$ are right angles and $\angle P = 40°$, we get that $\angle TOR = 140°$. Now OA and OB are the bisectors of $\angle SOT$ and $\angle SOR$ respectively. It follows that $\angle AOB = \frac{1}{2}\angle TOR = 70°$.

5. A function f satisfies the following conditions for all positive integers n.

$$f(2n) = f(n), \qquad f(2n+1) = f(n) + 1, \qquad f(1) = 1$$

Find the smallest n such that $f(n) = 7$.

Answer. $n = 127$.

Solution. Observe that $f(n)$ is the sum of the digits in the base 2 expansion of n. It follows that the smallest n with $f(n) = 7$ is the number 1111111_2, which is 127 in base 10.

6. From a two digit number N we subtract the number with the digits reversed and find that the result is a positive cube. Find all possible N.

Answer. $N = 30, 41, 52, 63, 74, 85, 96$.

Solution. Let $N = 10t + u$ for some decimal digits t and u. Then we have

$$10t + u - (10u + t) = 9(t - u).$$

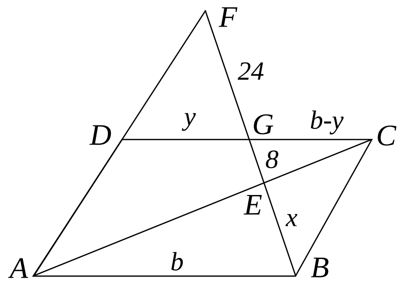
Since $t - u \leq 9$ we see that $9(t - u) \leq 81$. The only possible two digit cubes are

$$1, 8, 27, 64.$$

The only number which is divisible by 9 is 27, which implies $9(t - u) = 27$. Since $t = u + 3$, the number N must be of the form $N = 11u + 30$, where $10 \leq N \leq 99$. So $10 \leq 11u + 30 \leq 99$. This yields that $0 \leq u \leq 6$. Hence N must be one of

$$30, 41, 52, 63, 74, 85, 96.$$

7. A point F is taken on the extension of side AD of a parallelogram $ABCD$ as shown below. The segment BF intersects diagonal AC at E and the side DC at G. If $EF = 32$ and $GF = 24$, find BE.



Answer. $BE = 16$

Solution. Let $BE = x$, $DG = y$, and $AB = b$. Since $\triangle BEA \sim \triangle GEC$,

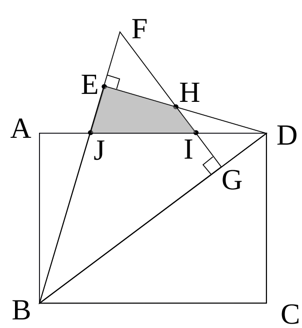
$$\frac{8}{x} = \frac{b-y}{b}, \quad b - y = \frac{8b}{x}, \quad y = b - \frac{8b}{x} = \frac{b(x-8)}{x}.$$

Since $\triangle FDG \sim \triangle BCG$,

$$\frac{24}{x+8} = \frac{y}{b-y}, \quad \frac{24}{x+8} = \frac{b(x-8)}{x \cdot (8b/x)} = \frac{x-8}{8}.$$

Therefore $x^2 - 64 = 192$ or $x = 16$.

8. Consider a rectangle $ABCD$ with $AB = 3$ and $BC = 4$. Reflect the right triangle $\triangle BCD$ along the diagonal BD to obtain a right triangle $\triangle BDE$, and then rotate $\triangle BCD$ about the vertex B to obtain a right triangle $\triangle BGF$. Let points H, I and J be the intersections between segments as below. Find the area of a quadrilateral $EJIH$.



Answer. The area is $\frac{49}{48}$.

Solution. Since $\triangle BCD \cong \triangle BED \cong \triangle BGF$,

$$BD = BF = 5, \quad ED = 3, \quad BE = BG = 4, \quad EF = DG = BD - BG = 1.$$

In addition, $\triangle JED \cong \triangle JAB$, which implies that $JD = AD - AJ = 4 - EJ$. By the Pythagorean theorem $EJ^2 + ED^2 = JD^2$, or

$$EJ^2 + 3^2 = (4 - EJ)^2.$$

Therefore, $EJ = AJ = \frac{7}{8}$. So $JD = 4 - AJ = \frac{25}{8}$. Let h denote the height of a right triangle $\triangle EJD$ with the base JD. Then we have

$$h = \frac{EJ \cdot ED}{JD} = \frac{7/8 \cdot 3}{25/8} = \frac{21}{25}.$$

The similarity between $\triangle BCD$ and $\triangle HGD$ implies that

$$\frac{DH}{DG} = \frac{5}{3} \quad \text{or} \quad DH = \frac{5}{3}.$$

If h' is the height of $\triangle IDH$ with the base ID then,

$$\frac{DH}{DE} = \frac{h'}{h} \quad \text{or} \quad h' = \frac{5/3 \cdot 21/25}{3} = \frac{21}{45}.$$

Since $\triangle DGI$ is similar to $\triangle BCD$,

$$\frac{ID}{DG} = \frac{5}{4} \quad \text{or} \quad ID = \frac{5}{4}.$$

Thus the desired area becomes

$$\text{Area}(EJIH) = \text{Area}(\triangle EJD) - \text{Area}(\triangle IDH) = \frac{1}{2}\left(\frac{7}{8} \cdot 3 - \frac{5}{4} \cdot \frac{21}{45}\right) = \frac{147}{144} = \frac{49}{48}.$$

9. If $x = \sqrt{3 - \sqrt{8}}$, find $x^7 + \frac{1}{x^7}$.

Answer. $x^7 + \frac{1}{x^7} = 338\sqrt{2}$

Solution. By rewriting $\sqrt{3 - \sqrt{8}} = \sqrt{3 - 2\sqrt{2}} = \sqrt{(\sqrt{2} - 1)^2}$ we have $x = \sqrt{2} - 1$. We also have

$$\begin{aligned}
x + \frac{1}{x} &= \sqrt{2} - 1 + \frac{1}{\sqrt{2} - 1} = \sqrt{2} - 1 + \sqrt{2} + 1 = 2\sqrt{2}, \\
x^2 + \frac{1}{x^2} &= \left(x + \frac{1}{x}\right)^2 - 2 = (2\sqrt{2})^2 - 2 = 6, \\
x^3 + \frac{1}{x^3} &= \left(x + \frac{1}{x}\right)^3 - 3\left(x + \frac{1}{x}\right) = (2\sqrt{2})^3 - 3 \cdot 2\sqrt{2} = 10\sqrt{2}, \\
x^4 + \frac{1}{x^4} &= \left(x^2 + \frac{1}{x^2}\right)^2 - 2 = 34.
\end{aligned}$$

So the given expression becomes

$$x^7 + \frac{1}{x^7} = \left(x^3 + \frac{1}{x^3}\right)\left(x^4 + \frac{1}{x^4}\right) - \left(x + \frac{1}{x}\right) = 340\sqrt{2} - 2\sqrt{2} = 338\sqrt{2}.$$

10. Find the number of all possible solutions of the equation $xyz = 8000$ when x, y and z are positive integers.

Answer. 280

Solution. Since $8000 = 2^6 5^3$ we may assume

$$x = 2^a 5^d, \quad y = 2^b 5^e, \quad z = 2^c 5^f$$

for some non-negative integers a, b, c, d, e, f. Thus

$$2^{a+b+c}\, 5^{d+e+f} = xyz = 8000 = 2^6 5^3.$$

We need to find the number of 6-tuples (a, b, c, d, e, f) with

$$a + b + c = 6, \quad d + e + f = 3.$$

Observe that, for any integer a with $0 \le a \le 6$, there exist $7 - a$ pairs (b, c) such that $b + c = 6 - a$. The number of triples (a, b, c) with $a + b + c = 6$ is

$$\sum_{a=0}^{6} (7 - a) = 7 + 6 + 5 + 4 + 3 + 2 + 1 = 28.$$

Similarly, the number of triples (d, e, f) with $d + e + f = 3$ is

$$\sum_{d=0}^{3} (4 - d) = 4 + 3 + 2 + 1 = 10.$$

Since we can choose such triples (a, b, c) and (d, e, f) independently the number of solutions is $28 \times 10 = 280$.

11. Let n be a three digit positive integer. Define a function $f(n)$ by

$f(n) =$ (the sum of the digits of n) $+$ (the sum of the products of two digits of n) $+$ (the product of the digits of n).

For example, if $n = 234$,

$$f(n) = (2 + 3 + 4) + (2 \cdot 3 + 3 \cdot 4 + 4 \cdot 2) + (2 \cdot 3 \cdot 4).$$

Find all possible three digit positive integers n such that $f(n) = n$.

Answer. $n = 199, 299, 399, 499, 599, 699, 799, 899, 999.$

Solution. We can write n as

$$n = 100a + 10b + c,$$

where $1 \le a \le 9$, $0 \le b, c \le 9$. Then

$$f(n) = a + b + c + ab + bc + ac + abc = (1 + a)(1 + b)(1 + c) - 1.$$

Thus we have

$$100a + 10b + c = (1 + a)(1 + b)(1 + c) - 1 \quad \text{or} \quad 100a + 10b + c = a(1 + b)(1 + c) + b + c + bc$$

We read the above as a linear expression in a and write

$$\big(100 - (1 + b)(1 + c)\big)a = b(c - 9).$$

Since $\big(100 - (1 + b)(1 + c)\big)a \ge 0$ and $b(c - 9) \le 0$ we have

$$\big(100 - (1 + b)(1 + c)\big)a = b(c - 9) = 0.$$

This implies $c = 9$ and then $b = 9$ because $a \ne 0$. So

$$n = 100a + 90 + 9 = (1 + a)(1 + 9)(1 + 9) - 1.$$

And since $1 \le a \le 9$, the desired integers are

$$n = 199, 299, 399, 499, 599, 699, 799, 899, 999.$$

12. Three radars are spaced 6, 8, and 10 miles from each other on the ground, which is assumed to be horizontal. The radars spot an airplane at a distance of 13 miles at the same time. What is the elevation of the airplane?

Answer. 12 miles.

Solution. Note that $(6, 8, 10)$ is a Pythagorean triple since $10^2 = 6^2+8^2$. This means that the location of the radars form a right triangle on the ground, and since the airplane is equidistant from the radars, it must be right above the intersection of the three perpendicular side bisectors of the triangle on the ground, which for a right triangle is at the middle point of its hypotenuse, call it H. If the location of the airplane is called P, then $\overline{PH}$ is perpendicular to the ground, and Pythagorean theorem implies that $PH = \sqrt{13^2 - (10/2)^2} = 12$.

13. Find all integer solutions (x, y) of the equation $15x^2 - 5xy - 16x + 7y + 6 = 0$.

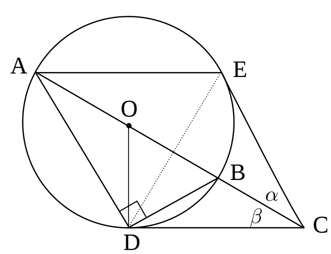
Answer. $(4, 14)$

Solution. We factor the given equation as follows.

$$\begin{aligned}
&15x^2 - 5xy - 16x + 7y + 6 = 0 && \Rightarrow && 5x(3x - y) - 16x + 7y + 6 = 0\\
&\Rightarrow 5x(3x - y) - 5(3x - y) - x + 2y + 6 = 0 && \Rightarrow && (3x - y)(5x - 5) - (3x - y) + 2x + y + 6 = 0\\
&\Rightarrow (3x - y)(5x - 6) - (3x - y) + 5x + 6 = 0 && \Rightarrow && (3x - y)(5x - 7) + 5x - 7 + 13 = 0\\
&\Rightarrow (3x - y + 1)(5x - 7) + 13 = 0 && \Rightarrow && (3x - y + 1)(5x - 7) = -13
\end{aligned}$$

Since we are looking for integer solutions and $5x - 7 \neq \pm 1$ for all integers x, we conclude that $5x - 7 = \pm 13$. Consequently $x = 4$. Thus we have $(12 - y + 1)13 = -13$ or $y = 14$.

14. Let AB be a diameter of a circle. A point C is chosen on the extension of AB beyond B. Points D and E are chosen on the circle so that $BC = BD$ and $EA = EC$. Find the ratio $BC : EC$ if CD is tangent to the circle.



Answer. $BC : EC = 1 : \sqrt{3}$

Solution. Let O be the center of the circle. Draw a line segment AD to have a right triangle $\triangle ABD$. Note that $\triangle BDC$ and $\triangle EAC$ are isosceles triangles. Also $\triangle OAD$ is an isosceles. Let $\angle OCE = \alpha$ and $\angle OCD = \beta$. Then we have $\angle OAE = \alpha$. Since $\angle ODC = 90°$,

$$\angle BDC = \angle ODA = \angle OAD = \beta.$$

Considering complement angles and supplementary angles we can show that

$$\angle AOD = 180° - 2\beta, \quad \angle DOC = 90° - \beta$$

and that

$$(180° - 2\beta) + (90° - \beta) = 180° \quad \text{or} \quad \beta = 30°.$$

Consequently $\triangle ODB$ is an equilateral triangle and so $OA = OD = OB = BD = BC$. Thus $\triangle OAD \cong \triangle BDC$. In particular, $AD = DC$.

Next draw a line segment DE to have $\triangle ADE \cong \triangle CDE$. This congruence implies that DE bisects $\angle ADC = 120°$. Now $\angle AED$ is the inscribed angle determined by an arc AD, and so $\angle AED = \dfrac{\angle AOD}{2} = 60°$. Therefore $\angle EAD = 60°$ and so $\triangle ADE$ is also an equilateral triangle.

The answer follows from $AD = DC = EC$;

$$BC : EC = BD : AD = 1 : \sqrt{3}.$$

15. Given a natural number n, four students A, B, C, and D claimed the following.

    A: $20 < n < 50$.

    B: n is a divisor of 120.

    C: n has 8 divisors (natural numbers)

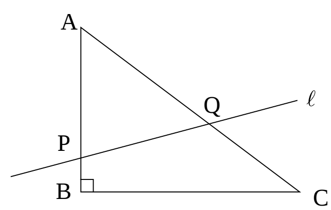
    D: n is a multiple of 12.

    If one and only one student made a false statement, who is it?

    Answer. D

    Solution. First consider the case when C and D are both true. The only multiple n of $12 = 2^2 \cdot 3$ with 8 divisors is $n = 2^3 \cdot 3 = 24$. However $n \neq 24$ as then all four are true. We see that either C or D is false. There are three integers $n = 24, 30, 40$ which satisfy both A and B. Since $n \neq 24$, we have $n = 30, 40$. For each of those numbers we see that D is false.

16. A line ℓ bisects both of the perimeter and the area of a right triangle $\triangle ABC$ as in the picture below. Find AQ if $AB = 3$, $BC = 4$.



Answer. $AQ = 3 + \dfrac{\sqrt{6}}{2}$.

Solution. Let $AP = y$ and $AQ = x$. Since the line bisects the perimeter of $\triangle ABC$ we see that $x + y = \dfrac{3+4+5}{2} = 6$. Let h be the height of $\triangle APQ$ with the base AP. Then we have

$$x : 5 = h : 4 \quad \text{or} \quad h = \frac{4x}{5}.$$

The area of $\triangle APQ$ is 3 because it is the half of the area of $\triangle ABC$, and so

$$\frac{yh}{2} = \frac{1}{2}\left(y\frac{4x}{5}\right) = \frac{2xy}{5} = 3.$$

Solving the system of equations $x + y = 6$ and $xy = \dfrac{15}{2}$ we have

$$x(6 - x) = \frac{15}{2} \quad \text{or} \quad 2x^2 - 12x + 15 = 0.$$

Thus we have

$$x = \frac{6 \pm \sqrt{6}}{2} = 3 \pm \frac{\sqrt{6}}{2}.$$

If $x = 3 - \dfrac{\sqrt{6}}{2}$ then $y = 3 + \dfrac{\sqrt{6}}{2} > 3 = AB$, which is impossible. So $AQ = 3 + \dfrac{\sqrt{6}}{2}$.

17. Find all pairs (x, y) satisfying the system $\begin{cases} 2x^2 + 7xy + 6y^2 = 12 \\ 7x^2 + 20xy + 14y^2 = 23. \end{cases}$

Answer. $(-1, 2)$ and $(1, -2)$

Solution. The left hand side of the first equation has a factoring;

$$2x^2 + 7xy + 6y^2 = 12 \Leftrightarrow (x + 2y)(2x + 3y) = 12.$$

By substituting $x + 2y = u$ and $2x + 3y = v$, we can rewrite the second equation as

$$7x^2 + 20xy + 14y^2 = 23 \Leftrightarrow 2v^2 - u^2 = 23.$$

The given system can be written as $\begin{cases} uv = 12 \\ 2v^2 - u^2 = 23. \end{cases}$ Eliminating $u = \dfrac{12}{v}$ we have

$$2v^2 - \left(\frac{12}{v}\right)^2 = 23 \Rightarrow 2v^4 - 23v^2 - 144 = 0 \Rightarrow (2v^2 + 9)(v^2 - 16) = 0.$$

So $v = \pm 4$ and thus $(u, v) = (3, 4)$ or $(-3, -4)$. This implies

$$\begin{cases} x + 2y = 3 \\ 2x + 3y = 4 \end{cases} \quad \text{or} \quad \begin{cases} x + 2y = -3 \\ 2x + 3y = -4. \end{cases}$$

The system has the solution

$$\begin{cases} x = -1 \\ y = 2 \end{cases} \quad \text{or} \quad \begin{cases} x = 1 \\ y = -2. \end{cases}$$

18. Let A and B be two positive integers and let

$$\begin{gathered} A + B = C \\ B + C = D \\ C + D = E \\ \vdots \\ L + M = N \\ \vdots \\ X + Y = Z. \end{gathered}$$

Find G if $A+B+C+\cdots+J=990$.

Answer. $G=90$.

Solution. With parameters $A=s$ and $B=t$ we can write

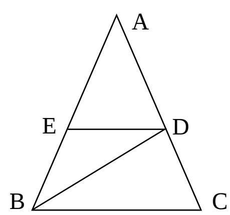
$$\begin{array}{lcrcr} A & = & s & & \\ B & = & & & t \\ C & = & s & + & t \\ D & = & s & + & 2t \\ E & = & 2s & + & 3t \\ F & = & 3s & + & 5t \\ G & = & 5s & + & 8t \\ H & = & 8s & + & 13t \\ I & = & 13s & + & 21t \\ J & = & 21s & + & 34t \end{array}$$

Adding up the above, we have

$$A+B+\cdots+J=55s+88t=11(5s+8t)=990.$$

Thus $G=5s+8t=90$.

19. An isosceles $\triangle ABC$ is made out of 3 smaller isosceles $\triangle AED$, $\triangle EBD$, and $\triangle BCD$ with $AE=AD$, $ED=EB$, $BD=BC$, and $AB=AC$. Find the area of $\triangle BCD$ if the area of $\triangle ABC$ is 1.



Answer. The area is $\dfrac{3-\sqrt{5}}{2}$.

Solution.

Let $\angle BCD=\alpha$ and $\angle BDE=\beta$. Then $\angle CBD=\pi-2\alpha$, $\angle AED=2\beta$, and $\angle ADE=\pi-(\alpha+\beta)$. Since $\triangle ADE$ and $\triangle ABC$ are isosceles,

$$2\beta=\pi-(\alpha+\beta) \quad \pi-2\alpha+\beta=\alpha.$$

So we have $\alpha=\dfrac{2\pi}{5}$ and $\beta=\dfrac{\pi}{5}$. This means that $\triangle DAB$ is an isosceles because $\angle BAD=\pi-2\alpha=\dfrac{\pi}{5}=\beta$. Let $AD=x$ and $DC=y$. Observe that two triangles $\triangle ABC$ and $\triangle BCD$ are similar. So we have

$$AC:BC=(x+y):x=x:y.$$

Since the area of $\triangle ABC$ is 1, the area of $\triangle BCD$ is the square of the ratio $\dfrac{x}{x+y}=\dfrac{y}{x}$. To find the ratio we rewrite it as

$$\frac{x}{y}=\frac{x+y}{x}=1+\frac{y}{x}.$$

Setting $X = \frac{y}{x}$ we have $X > 0$ and $X^2 + X - 1 = 0$, which implies $X = \frac{y}{x} = \frac{-1+\sqrt{5}}{2}$. Therefore the area of $\triangle BCD$ is $X^2 = \frac{3-\sqrt{5}}{2}$.

20. A triangle has sides $x^2 + x + 1$, $2x + 1$ and $x^2 - 1$. Find the largest interior angle of the triangle.

Answer. $120°$ or $\frac{2\pi}{3}$

Solution. All three sides have positive lengths; $2x + 1 > 0$ and $x^2 - 1 > 0$. So we have $x > 1$. Since

$$x^2 + x + 1 - (2x + 1) = x(x - 1) > 0, \quad x^2 + x + 1 - (x^2 - 1) = x + 2 > 0$$

we see that the longest side has the length $x^2 + x + 1$. Let θ be the angle opposite to it. Then by the Law of Cosine we get

$$\cos\theta = \frac{(2x+1)^2 + (x^2-1)^2 - (x^2+x+1)^2}{2(2x+1)(x^2-1)} = \frac{-(2x+1)(x^2-1)}{2(2x+1)(x^2-1)} = -\frac{1}{2}.$$

Therefore $\theta = 120°$.