2018 Best Student Exam Open
Texas A&M High School Students Contest
October 20, 2018

1. You purchase a stock and later sell it for \$144 per share. When you do, you notice that the percent increase was the same number as the original cost in dollars of each share. What was your original cost per share?

2. What is the coefficient of x^{13} after expanding and combining all like terms in the expression
$$(1 + x^3 + x^5)^{10}.$$

3. Numbers are placed in some points of a circle in the following way: in the first step, place 1's at the endpoints of a chosen diameter of the circle; these endpoints divide the circle into wto semicircular arcs. In the second step, at the midpoint of each of these semicircular arcs (i.e. at a point splitting the arc into two equal parts), we place the sum of the numbers at their endpoints. In the third step, at the midpoint of each of four arcs obtained in the previous step we place the sum of numbers at the endpoints of this arc and so on. Let S_n be the sum of all numbers that are written on a circle after making n steps. What is the ratio S_{2018}/S_{2017}?

4. One summer, the nations in the UK held a football (soccer) tournament where each team played every other team. The results are summarized below:

Team	Won	Lost	Draw	Goals For	Goals Against
England	3	0	0	7	1
Ireland	1	1	1	2	3
Wales	1	1	1	3	3
Scotland	0	3	0	1	6

If England beat Ireland by a score of 3-0, how many total goals were scored between England and Scotland?

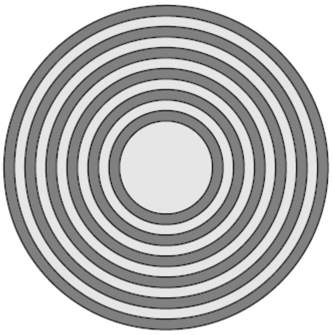
5. Find the minimal positive solution of the equation $\cos^{2019} x - \sin^{2019} x = 1$.

6. A single variable function f satisfies the following identity:
$$3f(x) + f(2 - x) = x^3$$
for every $x \in \mathbb{R}$. Find $f(3)$.

7. A differentiable single variable function f satisfies
$$f'(\sin^2 x) = \cos 2x + \tan^2 x \tag{1}$$
and $f(0) = 2$. Find $f\left(\frac{1}{2}\right)$.

8. Suppose f is a cubic polynomial with roots x, y, and z such that
$$x = \frac{1}{3 - yz}, \qquad y = \frac{1}{5 - zx}, \qquad z = \frac{1}{7 - xy}.$$
If $f(0) = 1$, compute $f(xyz + 1)$.

9. Dean and Deanna are throwing darts at the dartboard shown below, with alternating dark and light regions delineated by concentric circles. The smallest circle (encircling the light region in the center) has radius r, and each larger circle has radius 1 more than the next smaller circle, up to the largest circle (encircling the entire dartboard) which has radius $r + 11$. Dean scores a point if he throws a dart into a dark region, and Deanna scores a point if she throws a dart into a light region. Assuming a dart will hit every point of the dartboard with equal probability, for which value of r is the game fair (that is, both players are equally likely to score)?



10. What is the 100th digit to the right of the decimal point of decimal expansion of $(1+\sqrt{2})^{500}$?

11. You alternate tossing two weighted coins. The first coin you toss has a $\frac{2}{3}$ chance of landing heads and a $\frac{1}{3}$ chance of landing tails; the second coin has a $\frac{1}{4}$ chance of landing heads and a $\frac{3}{4}$ chance of landing tails. What is the probability that you will toss two heads in a row before you toss two tails in a row?

12. A plane P in R^3 is said to be a *symmetry plane* for a set $S \subset \mathrm{R}^3$ if the image of S under the mirror reflection with respect to the plane P is equal to S (in other words, the mirror reflection preserves S). Assume that S is the union of three nonparallel lines. What maximal possible number of symmetry planes may such set S have?

13. Given a 2×2 square, let A be the set of all points whose distance to the center of the square and to the nearest edge of the square are equal. What is the area enclosed by A? Express your answer in the form $a + b\sqrt{c}$, where a and b are rational numbers and c is an integer.

14. Let D and E be points on the unit circle corresponding to the angles $\frac{2\pi}{7}$ and $\frac{\pi}{11}$ respectively. Define S to be the area of the region below $\overset{\frown}{DE}$ and above the x-axis and T to be the area of the region to the left of $\overset{\frown}{DE}$ and right of the y-axis. What is $S + T$?

15. Evaluate

$$\lim_{x\to 0} \frac{\tan\Big(\tan(3x)\Big) - \sin\Big(\sin(3x)\Big)}{\tan(2x) - \sin(2x)}$$

16. Nir has a blackboard that initially has the number 1 written on it. He repeatedly takes the most recent number written on the blackboard, adds either 1, 2, or 3 to it, and writes that number next to the existing numbers. He stops as soon as he writes a number greater than or equal to 100. At the end, Nir notices remarkably that there are no multiples of 3 or 4 written on the blackboard. How many different possible sequences of numbers could be on the blackboard?

17. Consider all quadruples (x, y, z, w) satisfying the following system of equations:

$$\begin{cases} x^2 + y^2 = 9 \\ z^2 + w^2 = 25 \\ xw + yz = 15 \end{cases}$$

What is the maximal possible value of $x + z$ for such quadruples?

18. Keyu randomly picks two real numbers from the interval $(0, 3)$. Paawan randomly picks one real number from the interval $(1, 2)$. If all numbers are chosen independently, what is the probability that Paawan's number is strictly between Keyu's two numbers?

19. Let $a_1, a_2, a_3, \dots$ be a sequence of positive integer numbers such that $a_i \leq a_{i+1}$ and $a_{a_k} = 3k$ for every k. Find a_{110}.

20. Consider four points in $\mathbb{R}^3$ that do not lie in the same plane. How many different parallelepipeds have vertices at these points?