2018 Best Student Exam Solutions
Texas A&M High School Students Contest
October 20, 2018

1. You purchase a stock and later sell it for \$144 per share. When you do, you notice that the percent increase was the same number as the original cost in dollars of each share. What was your original cost per share?

   Answer $\boxed{\$80}$.

   Solution. Let x be the original cost of the stock. Then $\left(1+\dfrac{x}{100}\right)x = 144$, or $x^2+100x-14400=0$. Then $\dfrac{D}{4} = 50^2 - (-14400) = 2500 + 14400 = 16900 = 130^2 \Rightarrow x_{1,2} = -50 \pm 130$. The only positive solution is $x = \boxed{\$80}$ per share.

2. What is the coefficient of x^{13} after expanding and combining all like terms in the expression

$$(1 + x^3 + x^5)^{10}.$$

   Answer: $360 = \dfrac{10!}{7!2!}$.

   Solution. Note that the only solution of the equation $3k + 5l = 13$ in nonnegative integers is $k = 1$, $l = 2$ and in this case $k + l < 10$. So the only way to obtain the term x^{13}, when opening the brackets of $(1 + x^3 + x^5)^{10}$ is to pick up the term x^3 from one of the factors , the term x^5 from 2 of the factors and the term 1 from the rest, i.e. 7, factors. The number of ways to do this is equal to $\dfrac{10!}{1!2!7!} = \dfrac{8 \cdot 9 \cdot 10}{2} = \boxed{360}$.

3. Numbers are placed in some points of a circle in the following way: in the first step, place 1's at the endpoints of a chosen diameter of the circle; these endpoints divide the circle into two semicircular arcs. In the second step, at the midpoint of each of these semicircular arcs (i.e. at a point splitting the arc into two equal parts), we place the sum of the numbers at their endpoints. In the third step, at the midpoint of each of four arcs obtained in the previous step we place the sum of numbers at the endpoints of this arc and so on. Let S_n be the sum of all numbers that are written on a circle after making n steps. What is the ratio S_{2018}/S_{2017}?

   Answer $\boxed{3}$

   Solution. Since each point of the circle marked in the first nth steps is the endpoint of exactly two arcs, each number constructed in the first n steps is used twice in the construction of the numbers in the $(n + 1)$st step. So the sum of all numbers , constructed in the $(n + 1)$st step is twice of the sum of all numbers constructed in the first n steps. i.e. is equal to $2S_n$. Therefore S_{n+1}, being by definition the sum of all numbers constructed in the first $n + 1$ steps, is equal to $3S_n$. So, $S_{n+1}/S_n = \boxed{3}$ for any n.

4. One summer, the nations in the UK held a football (soccer) tournament where each team played every other team. The results are summarized below:

Team	Won	Lost	Draw	Goals For	Goals Against
England	3	0	0	7	1
Ireland	1	1	1	2	3
Wales	1	1	1	3	3
Scotland	0	3	0	1	6

If England beat Ireland by a score of 3-0, how many total goals were scored between England and Scotland?

Answer $\boxed{2}$.

Solution. Removing the England-Ireland result yields the following updated table

Team	Won	Lost	Draw	Goals For	Goals Against
England	2	0	0	4	1
Ireland	1	0	1	2	0
Wales	1	1	1	3	3
Scotland	0	3	0	1	6

So Ireland had to draw 0-0 with Wales and beat Scotland 2-0. This means the scores in the remaining Wales matches had to either both be 3-0 or both be 2-1 (one win, one loss). If they were 3-0, England would not have won their third match (1-1), so Wales beat Scotland 2-1. This means England beat Scotland 2-0, for a total of $\boxed{2}$ goals.

5. Find the minimal positive solution of the equation

$$\cos^{2019} x - \sin^{2019} x = 1. \tag{1}$$

Answer $\boxed{\dfrac{3\pi}{2}}$.

Solution. We have the following chain of inequalities

$$\cos^{2019} x - \sin^{2019} x \le |\cos^{2019} x - \sin^{2019} x| \le |\cos^{2019} x| + |\sin^{2019} x| \le \cos^2 x + \sin^2 x = 1, \tag{2}$$

where we used the triangle inequality for the second inequality and the fact that $|\cos x| \le 1$ and $|\sin x| \le 1$ for the third inequality. So, x is a solution of (1) if and only if all inequalities in (2) are actually the equalities.

The first inequality in the chain (2) is the equality if and only if $\cos^{2019} x \ge \sin^{2019} x$. The first two inequalities in the chain (2) are equalities if $\cos^{2019} x \ge 0$ and $\sin^{2019} x \le 0$, and all three inequalities in (2) are equalities if and only if either $\cos x = 1$ and $\sin x = 0$ or $\cos x = 0$ and $\sin x = -1$. The first case implies that $x = 2\pi n$, $n \in \mathbb{Z}$ and the second case implies that $x = -\dfrac{\pi}{2} + 2\pi n$, $n \in \mathbb{Z}$. Hence, the minimal positive solution is $x = \boxed{\dfrac{3\pi}{2}}$.

6. A single variable function f satisfies the following identity:

$$3f(x) + f(2 - x) = x^3 \tag{3}$$

for every $x \in \mathbb{R}$. Find $f(3)$.

Answer: $\boxed{\dfrac{82}{8}}$.

Solution. Plug $2-x$ instead of x into (3) to get

$$3f(2-x) + f(x) = (2-x)^3. \tag{4}$$

This gives us the system of two linear equations in $f(x)$ and $f(2-x)$:

$$\begin{cases} 3f(x) + f(2-x) = x^3 \\ f(x) + 3f(2-x) = (2-x)^3 \end{cases}$$

Eliminate $f(2-x)$ by subtracting the second equation from the tripled first equation. Then we get $8f(x) = 3x^3 - (2-x)^3$, i.e.

$$f(x) = \frac{1}{8}\big(3x^3 - (2-x)^3\big)$$

This function satisfies (3), as can be verified by the direct substitution into (3). So, in fact we found the unique function satisfying (3). Finally,

$$f(3) = \frac{1}{8}(3 \cdot 3^3 - (-1)^3) = \boxed{\frac{82}{8}}$$

7. A differentiable single variable function f satisfies

$$f'(\sin^2 x) = \cos 2x + \tan^2 x \tag{5}$$

and $f(0) = 2$. Find $f\big(\frac{1}{2}\big)$.

Answer $\boxed{-\ln\dfrac{1}{2} + \dfrac{7}{4} = \ln 2 + \dfrac{7}{4}}$.

Solution. Note that $\cos 2x = 1 - 2\sin^2 x$ and $\tan^2 x = \dfrac{\sin^2 x}{1-\sin^2 x}$. Using this and substituting $y = \sin^2 x$ into (5) we get

$$f'(y) = 1 - 2y + \frac{y}{1-y} = \frac{1}{1-y} - 2y.$$

Integrating both parts we have $f(y) = -\ln|1-y| - y^2 + C$ for some constant C. The condition $f(0) = 2$ implies that $C = 2$. Hence $f(y) = -\ln|1-y| - y^2 + 2$. So,

$$f\big(\frac{1}{2}\big) = -\ln\frac{1}{2} - \frac{1}{4} + 2 = \boxed{-\ln\frac{1}{2} + \frac{7}{4} = \ln 2 + \frac{7}{4}}.$$

8. Suppose f is a cubic polynomial with roots x, y, and z such that

$$x = \frac{1}{3-yz}, \qquad y = \frac{1}{5-zx}, \qquad z = \frac{1}{7-xy}.$$

If $f(0) = 1$, compute $f(xyz + 1)$.

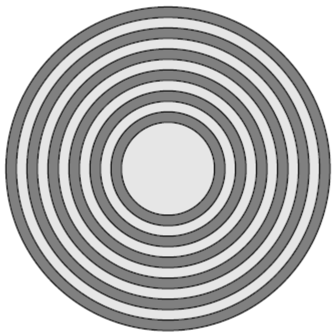
Answer: $\boxed{-48}$.

Solution. Let a be the leading coefficient of f, so that $f(r) = a(r - x)(r - y)(r - z)$. Then $f(0) = -axyz = 1$. From the three other equations given, we can rearrange to get

$$xyz + 1 = 3x = 5y = 7z$$

and thus $f(xyz + 1) = a(xyz + 1 - x)(xyz + 1 - y)(xyz + 1 - z) = a(3x - x)(5y - y)(7z - z) = axyz \cdot 2 \cdot 4 \cdot 6 = \boxed{-48}$.

9. Dean and Deanna are throwing darts at the dartboard shown below, with alternating dark and light regions delineated by concentric circles. The smallest circle (encircling the light region in the center) has radius r, and each larger circle has radius 1 more than the next smaller circle, up to the largest circle (encircling the entire dartboard) which has radius $r + 11$. Dean scores a point if he throws a dart into a dark region, and Deanna scores a point if she throws a dart into a light region. Assuming a dart will hit every point of the dartboard with equal probability, for which value of r is the game fair (that is, both players are equally likely to score)?



Answer: $\boxed{1 + \sqrt{12} = 1 + 2\sqrt{3}}$.

Solution. The area of the entire dartboard is $\pi(r + 11)^2$, and the area of the six dark rings is $\pi\left(-r^2 + (r + 1)^2 - (r + 2)^2 + \cdots + (r + 11)^2\right)$. We simplify the expression on the left, observing that the r^2 terms cancel each other out:

$$\pi\left(-r^2 + (r + 1)^2 - (r + 2)^2 + \cdots + (r + 11)^2\right) = \pi\left[(-0r + 2r - 4r + 6r - ... + 22r) +\right.$$

$$\left.(-0^2 + 1^2 - 2^2 + \cdots + 11^2)\right] = \pi\left[6 \cdot 2r + (1 + 5 + 9 + 13 + 17 + 21)\right] = \pi(12r + 66) = \frac{1}{2}\pi(r + 11)^2$$

Therefore, the problem is reduced to solving the equation: $\pi(12r+66) = \frac{1}{2}\pi(r+11)^2$. So $24r+132 = r^2 + 22r + 121 \implies r^2 - 2r - 11 = 0$. Then $r = \frac{2\pm\sqrt{4+44}}{2} = 1 \pm \sqrt{12} = 1 \pm 2\sqrt{3}$. The radius is obviously positive, so the answer is $\boxed{1 + \sqrt{12} = 1 + 2\sqrt{3}}$.

(In general, if there are N dark rings, we have $r = 1 + \sqrt{2N}$. This problem is the $N = 6$ case.

10. What is the 100th digit to the right of the decimal point of decimal expansion of $(1 + \sqrt{2})^{500}$?

Answer $\boxed{9}$.

Solution. $(1+\sqrt{2})^{500}+(1-\sqrt{2})^{500}$ is a positive integer since, in both expansions, all odd powers of $\sqrt{2}$ have opposite signs. Further, $|1-\sqrt{2}|^{500} < \left(\frac{1}{2}\right)^{500} < \left(\frac{1}{1000}\right)^{50}$, or 10^{-150} (using the fact that $2^{10} > 10^3$). Subtracting this number from our positive integer above yields a decimal with 150 leading 9's, so the 100th digit is $\boxed{9}$.

11. You alternate tossing two weighted coins. The first coin you toss has a $\frac{2}{3}$ chance of landing heads and a $\frac{1}{3}$ chance of landing tails; the second coin has a $\frac{1}{4}$ chance of landing heads and a $\frac{3}{4}$ chance of landing tails. What is the probability that you will toss two heads in a row before you toss two tails in a row?

Answer $\boxed{\dfrac{13}{33}}$

Solution. We will describe the result of tossing by the words consisting of letters H and T with H and T standing for heads and tails, respectively. Then HH comes before TT if and only if the corresponding words are either

$$\underbrace{HTHT\dots HT}_{HT \text{ repeated } k \text{ times}}\ HH\dots \tag{6}$$

or

$$\underbrace{THTH\dots TH}_{TH \text{ repeated } k \text{ times}}\ THH\dots \tag{7}$$

for some $k \geq 0$. The probability to get the word in (6) is $\left(\frac{2}{3}\times\frac{3}{4}\right)^k \frac{2}{3}\times\frac{1}{4} = \left(\frac{1}{2}\right)^k \frac{1}{6}$. The probability to get the word in (7) is $\left(\frac{1}{3}\times\frac{1}{4}\right)^k \frac{1}{3}\times\frac{1}{4}\times\frac{2}{3} = \left(\frac{1}{12}\right)^k \frac{1}{18}$. Then the total probability of the required event is the sum of these probabilities over all nonnegative k, which can be computed using the formula for the sum of the geometric progression:

$$\frac{1}{6}\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k + \sum_{k=0}^{\infty}\frac{1}{18}\left(\frac{1}{12}\right)^k = \frac{1}{6}\times\frac{1}{1-\frac{1}{2}} + \frac{1}{18}\times\frac{1}{1-\frac{1}{12}} = \frac{1}{3}+\frac{2}{33} = \boxed{\frac{13}{33}}$$

.

12. A plane P in R^3 is said to be a *symmetry plane* for a set $S \subset \mathrm{R}^3$ if the image of S under the mirror reflection with respect to the plane P is equal to S (in other words, the mirror reflection preserves S). Assume that S is the union of three lines such that there are two nonparallel lines among them. What maximal possible number of symmetry planes may such set S have?

Answer $\boxed{9}$.

Solution. First note that if the three lines are all parallel then every plane perpendicular to them is a plane of symmetry, so the assumption that there is a pair of nonparallel lines among them is important to rule out the answer ∞. Now consider separately the following two cases:

(a) *Assume that all three lines lie in the the same plane*, denoted by P. Then obviously P is a plane of symmetry. Since any mirror reflection, preserving the union of three lines, has to preserve

P, any other plane of symmetry has to be perpendicular to P. Hence we can consider the restriction of the corresponding mirror reflection to P so that it is the planar mirror reflection with respect to the line in the intersection of P with the reflecting plane and the problem becomes planar:

To describe the maximal number of lines of symmetry for three lines on a plane, if there are two nonparallel lines among them.

Let us solve this planar problem. If a planar mirror reflection sends the union of three lines with the property above to itself, then it must swap two lines and preserve one line (otherwise all three lines are parallel, which is not allowed). Therefore the reflecting line must either coincide with one of the lines or to be perpendicular to one of the lines. In the latter case this perpendicular line is uniquely defined as follows: if the other two lines are intersecting, then it must pass through their intersection point and if they are parallel, then it must be the equidistant parallel line to them. This shows that to each line in our family of three lines corresponds at most 2 lines of symmetries, so the total number of lines of symmetries does not exceeds $3 \times 2 = 6$. Note that the upper bound 6 is achieved, if we take three lines in the plane intersecting in one point and dividing the plane into six sectors of equal angle $\dfrac{\pi}{3}$.

Hence , returning to the problem in $\mathbb{R}^3$, the maximal number of the plane of symmetries in the considered case is 7 (because we also have to count the plane P itself).

(b) *Assume that three lines are not in the same plane.* By the argument similar to above, if a mirror reflection sends the union of three lines to itself, then it either swap two lines and preserves one line or preserve all three lines. In both cases at least one line is preserved. A line l_1 is preserved by a mirror reflection if and only if either the reflecting plane P contains l_1 or it is perpendicular to l_1 .Consider this cases separately:

i. *If P contains l_1* , then by assumption either P does not contain only one of remaining line or it does not contain both of remaining lines. Consider these subcases separately:

A. *If P does not contain one of the remaining lines*, say l_2, then this line must be orthogonal to P, so P is uniquely determined by l_1 and l_2.

B. *If P does not contain both of remaining lines*, say l_2 and l_3 then l_2 and l_3 lie in the same plane $\widetilde{P}$ perpendicular to P such that l_1 does not lie in $\widetilde{P}$ (by our assumption) and P uniquely determined by l_1 and $\widetilde{P}$ (it passes through l_1 and $P \cap \widetilde{P}$).

ii. *If P is perpendicular to l_1*, then consider the following cases for the other lines l_2 and l_3:

A. *If l_2 and l_3 intersect*, then P must pass through there point of intersection, so P is uniquely determined by l_1, l_2, and l_3;

B. *If l_2 and l_3 are parallel* , then P is not perpendicular to them, otherwise all three lines are parallel, which is not allowed. So P must be equidistant and parallel to them. Again P is uniquely determined by l_1, l_2, and l_3;

C. *If l_2 and l_3 are skew*, they cannot be preserved or swapped by a mirror reflection, so this case is impossible.

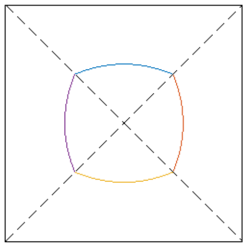
In summary, any mirror symmetry preserves a line and if it preserves a line, then there is at most three possibilities for the plane of symmetries, described by iA, iB, and ii. So, there is at most $3 \times 3 = 9$ plane of symmetries. This upper bound is achieved, if we take three lines passing through one point and perpendicular one to each other, for example, three axes of a Cartesian coordinate system (x, y, z): the planes of symmetries are given by the following planes: $x = 0, y = 0, z = 0, x = y, x = -y, z = x, z = -x, z = y$, and $z = -y$.

Comparing the answers in (a) and (b) we get $\boxed{9}$ as the answer.

13. Given a 2×2 square, let A be the set of all points whose distance to the center of the square and to the nearest edge of the square are equal. What is the area enclosed by A? Express your answer in the form $a + b\sqrt{c}$, where a and b are rational numbers and c is an integer.

Answer $\boxed{-\frac{20}{3} + \frac{16}{3}\sqrt{2}}$.

Solution. Place the origin at the center of the square so all corners have coordinates $(\pm 1, \pm 1)$. Choose a point $(x, y) \in A$ which lies above the curve $y = |x|$. Then $(x, 1)$ is the closest edge point, so we have $\sqrt{x^2 + y^2} = (1 - y)$, which can be rewritten as $y = \frac{1 - x^2}{2}$. By symmetry, the area inside A is eight times the area between this curve and $y = x$ in the first quadrant (see figure below).



$\frac{1 - x^2}{2} = x$ when $x = -1 + \sqrt{2}$. So the area is

$$8 \int_0^{-1+\sqrt{2}} \left(\frac{1 - x^2}{2} - x \right) dx$$

$$= 4 \left(x - \frac{1}{3} x^3 - x^2 \right) \Big|_0^{-1+\sqrt{2}}$$

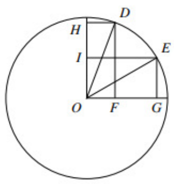
$$= 4 \left((-1 + \sqrt{2}) - \frac{1}{3}(-1 + 3\sqrt{2} - 6 + 2\sqrt{2}) - (1 - 2\sqrt{2} + 2) \right)$$

$$= \boxed{-\frac{20}{3} + \frac{16}{3}\sqrt{2}}$$

14. Let D and E be points on the unit circle corresponding to the angles $\frac{2\pi}{7}$ and $\frac{\pi}{11}$ respectively. Define S to be the region below $\overset{\frown}{DE}$ and above the x-axis and T to be the region to the left of $\overset{\frown}{DE}$ and right of the y-axis. What is $S + T$?

Answer $\boxed{\frac{15\pi}{77}}$

Solution. Given the region below, S is the area of region $DEGF$, T is the area of region $DEIH$, and C is the area of sector ODE. Then $S = C + \text{area}(\triangle OEG) - \text{area}(\triangle ODF)$ and $T = C + \text{area}(\triangle ODH) - \text{area}(\triangle OEI)$. Clearly, $\triangle OEG \cong \triangle OEI$ and $\triangle ODF \cong \triangle ODH$, so $S + T = 2C$ (NOTE this is true independent of our choices of D and E!). So $S + T = 2\left(\frac{1}{2}(1)^2\left(\frac{2\pi}{7} - \frac{\pi}{11}\right)\right) = \boxed{\frac{15\pi}{77}}$.



15. Evaluate

$$\lim_{x\to 0} \frac{\tan\Big(\tan(3x)\Big) - \sin\Big(\sin(3x)\Big)}{\tan(2x) - \sin(2x)} \tag{8}$$

Answer $\boxed{\frac{27}{4}}$.

Solution. Recall that the Maclaurin expansion of $\sin x$ and $\tan x$ up to the terms of order x^3 are

$$\sin x = x - \frac{x^3}{6} + o(x^3), \tag{9}$$

$$\tan x = x + \frac{x^3}{3} + o(x^3) \tag{10}$$

Hence,

$$\tan(2x) - \sin(2x) = \left(\frac{8}{3} + \frac{8}{6}\right) x^3 + o(x^3) = 4x^3 + o(x^3). \tag{11}$$

Further,

$$\begin{aligned}&\tan\Big(\tan(3x)\Big) = \tan\Big(3x + \frac{3^3x^3}{3} + o(x^3)\Big) = \tan\Big(3x + 9x^3 + o(x^3)\Big) = \\ &3x + 9x^3 + \frac{1}{3}(3x)^3 + o(x^3) = 3x + 18x^3 + o(x^3);\end{aligned} \tag{12}$$

$$\begin{aligned}&\sin\Big(\sin(3x)\Big) = \sin\Big(3x - \frac{3^3x^3}{6} + o(x^3)\Big) = \sin\Big(3x - \frac{9}{2}x^3 + o(x^3)\Big) = \\ &3x - \frac{9}{2}x^3 - \frac{1}{6}(3x)^3 + o(x^3) = 3x - 9x^3 + o(x^3);\end{aligned} \tag{13}$$

Consequently,

$$\tan\Big(\tan(3x)\Big) - \sin\Big(\sin(3x)\Big) = 27x^3 + o(x^3). \tag{14}$$

Using (11) and (14) we get

$$\frac{\tan\Big(\tan(3x)\Big) - \sin\Big(\sin(3x)\Big)}{\tan(2x) - \sin(2x)} = \frac{27x^3 + o(x^3)}{4x^3 + o(x^3)} = \boxed{\frac{27}{4}}.$$

16. Nir has a blackboard that initially has the number 1 written on it. He repeatedly takes the most recent number written on the blackboard, adds either 1, 2, or 3 to it, and writes that number next to the existing numbers. He stops as soon as he writes a number greater than or equal to 100. At the end, Nir notices remarkably that there are no multiples of 3 or 4 written on the blackboard. How many different possible sequences of numbers could be on the blackboard?

    Answer: $\boxed{3^8 = 6561}$

    Solution. Modulo 12, the only numbers Nir can write on the board are $1, 2, 5, 7, 10, 11$. Furthermore, with the exception of 11 and 1, he cannot skip any of these numbers, since the distance between the previous and the next number is greater than 3. So we see that the first number he writes after the 1 is 2, and then we count how many ways there are for him to get from one number congruent to 2 (mod 12) to the next one. Since only 11 and 1 can be skipped, but not both of them, there are only the following three ways:

    (a) nothing is skipped: $2, 5, 7, 10, 11, 1, 2$;
    (b) only 11 is skipped: $2, 5, 7, 10, 1, 2$;
    (c) only 1 is skipped: $2, 5, 7, 10, 11, 2$.

    Furthermore, Nir does this eight times, to get from $12 \cdot 0 + 2$ to $12 \cdot 8 + 2 = 98$. (Once he writes 98, the next number he writes will be 101 and then he will stop.) So the answer is $\boxed{3^8 = 6561}$.

17. Consider all quadruples (x, y, z, w) satisfying the following system of equations:

$$\begin{cases} x^2 + y^2 = 9 \\ z^2 + w^2 = 25 \\ xw + yz = 15 \end{cases} \tag{15}$$

    What is the maximal possible value of $x + z$ for such quadruples?

    Answer: $\boxed{\sqrt{34}}$.

    Solution. The system (15) can be interpreted in terms of the magnitudes and the dot product of vectors $\mathbf{a} = (x, y)$ and $\mathbf{b} = (w, z)$. Indeed, (15) is equivalent to

$$\begin{cases} |\mathbf{a}| = 3 \\ |\mathbf{b}| = 5 \\ \mathbf{a} \cdot \mathbf{b} = 15 \end{cases} \tag{16}$$

    Note that $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\phi$, where ϕ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$. Hence, by (16), $15 = 15\cos\phi$. Therefore vectors $\mathbf{a}$ and $\mathbf{b}$ are collinear. Consequently, $\mathbf{a} = 3(\cos\theta, \sin\theta)$ and $\mathbf{b} = 5(\cos\theta, \sin\theta)$ for some angle θ or, equivalently, $x = 3\cos\theta, y = 3\sin\theta, w = 5\cos\theta, z = 5\sin\theta$. Now we have to find θ such that $x + z = 3\cos\theta + 5\sin\theta$ is maximal possible. Using the property of the dot product again (namely the Cauchy-Schwarz inequality) we get that

$$3\cos\theta + 5\sin\theta \leq \sqrt{3^2 + 5^2}\sqrt{\cos^2\theta + \sin^2\theta} = \sqrt{34}$$

and the equality is achieved if and only if the vectors $(3, 5)$ and $(\cos\theta, \sin\theta)$ are collinear, or equivalently when $\cos\theta = \dfrac{3}{\sqrt{34}}$ and $\sin\theta = \dfrac{5}{\sqrt{34}}$. Therefore, the maximal possible $x + z$ is equal to $\boxed{\sqrt{34}}$.

18. Keyu randomly picks two real numbers from the interval $(0, 3)$. Paawan randomly picks one real number from the interval $(1, 2)$. If all numbers are chosen independently, what is the probability that Paawan's number is strictly between Keyu's two numbers?

    Answer $\boxed{\dfrac{13}{27}}$.

    Solution. Let k_1 and k_2 be Keyu's numbers and p be Paawan's number. There are nine possibilities for the ordered pair $(\lfloor k_1 \rfloor, \lfloor k_2 \rfloor)$, where $\lfloor k \rfloor$ denotes the maximal integer number not greater than k.

    First, if $\lfloor k_1 \rfloor = \lfloor k_2 \rfloor = 1$, then the probability that Paawan's number is between the two is $\frac{1}{3}$, since if we randomly choose the three numbers from $(1, 2)$, there is a $\frac{1}{3}$ chance that the middle one is Paawan's. (We neglect the possibility of any number being *exactly* an integer; the probability of this goes to zero since the distribution is continuous.)

    If exactly one of $(\lfloor k_1 \rfloor, \lfloor k_2 \rfloor)$ is 1, then p must be either greater or less than that value (WLOG k_1), independent of the value of k_2; thus the probability in this case is $\frac{1}{2}$. (As a specific example, suppose $\lfloor k_1 \rfloor = 1$ and $\lfloor k_2 \rfloor = 0$. Then $p > k_2$, so we must have $p < k_1$ and the probability is thus $k_1 - 1$. The average value of this as k_1 goes from 1 to 2 is $\frac{1}{2}$. The other three cases are similar.)

    If neither $\lfloor k_1 \rfloor$ nor $\lfloor k_2 \rfloor$ are equal to 1, but $\lfloor k_1 \rfloor = \lfloor k_2 \rfloor$, then p cannot possibly be between them and the probability is zero. However, if $\lfloor k_1 \rfloor \neq \lfloor k_2 \rfloor$ then p will be between them, with probability 1.

    As a visual aid, the probabilities for each of these nine pairs are displayed in the grid below:

2	1	$\frac{1}{2}$	0
1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$
$\lfloor k_2 \rfloor = 0$	0	$\frac{1}{2}$	1
	$\lfloor k_1 \rfloor = 0$	1	2

    The average probability, then is $\frac{1}{9}\left(1 + \frac{1}{2} + 0 + \frac{1}{2} + \frac{1}{3} + \frac{1}{2} + 0 + \frac{1}{2} + 1\right) = \frac{1}{9} \cdot \frac{13}{3} = \boxed{\dfrac{13}{27}}$.

19. Let $a_1, a_2, a_3, \ldots$ be a sequence of positive integer numbers such that $a_i \leq a_{i+1}$ and $a_{a_k} = 3k$ for every k. Find a_{110}.

    Answer: $\boxed{191}$

    Solution. First note that the sequence $\{a_k\}$ is strictly increasing, i.e. $a_k < a_{k+1}$ for every k. Indeed, if there exists $a_k = a_{k+1} = n$, then, on one hand, $a_n = a_{a_k} = 3k$ and, on the other hand, $a_n = a_{a_{k+1}} = 3(k + 1)$, so we get a contradiction.

Besides, $a_1 > 1$, otherwise, if $a_1 = 1$, then $a_1 = a_{a_1} = 3$, which leads to a contradiction.

Further, from the strict monotonicity, it follows that $a_1 < a_{a_1} = 3$. So, $a_1 = 2$.

Hence $a_2 = a_{a_1} = 3$, $a_3 = a_{a_2} = 6$, $a_6 = a_{a_3} = 9$, $a_9 = a_{a_6} = 18$, $a_{18} = a_{a_9} = 27$, $a_{27} = a_{a_{18}} = 54$, $a_{54} = a_{a_{27}} = 81$, $a_{81} = a_{a_{54}} = 162$, $a_{162} = a_{a_{81}} = 243$. Note that

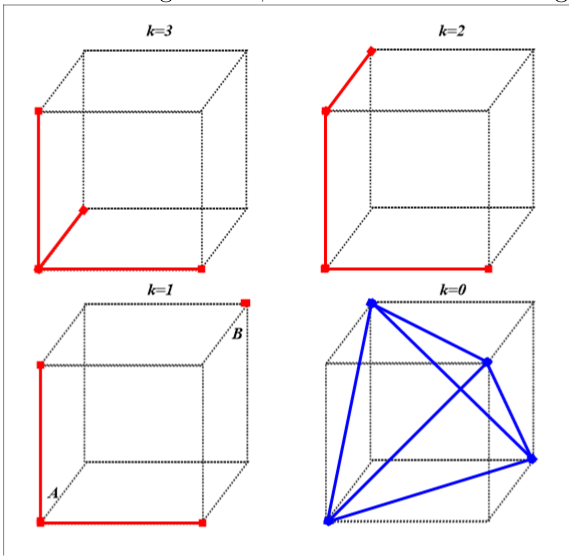
$$a_{162} - a_{81} = 243 - 162 = 162 - 81 = 81.$$

Hence from the strong monotonicity $a_{82}, a_{83}, \ldots, a_{162}$ are successive integers from 163 to 243 or, equivalently, $a_{81+k} = 162 + k$ for all $1 \leq k \leq 81$. In particular, $a_{110} = a_{81+29} = 162 + 29 = \boxed{191}$.

20. Consider four points in $\mathbb{R}^3$ that do not lie in the same plane. How many different parallelepipeds have vertices at these points?

Answer $\boxed{29}$.

Solution. First fix a parallelepiped. To any choice of four non-coplanar vertices of a parallelepiped assign the number of faces containing 3 chosen vertices. We call such faces *rigid* with respect to this choice (since they are uniquely reconstructed from the choice). The number of rigid faces does not exceed 3. Otherwise there is two parallel rigid faces, each of which contains 3 chosen vertices, which contradicts the fact that the total number of the chosen vertices is 4. Beside, for every integer k between 0 and 3 there exist a choice of 4 vertices with k rigid faces, as shown in the following figure:



Now, given four non-coplanar points, for every integer k between 0 and 3 calculate the number of parallelepipeds such that these four points are vertices of them and the number of rigid faces with respect to these vertices is equal to k:

1. **Case $k = 3$.** In this case the rigid faces have exactly one common vertex and the parallelepiped is uniquely defined by this choice. Since any of the four points can be chosen as this common vertex we have 4 parallelepipeds in this case.

2. **Case $k = 2$.** In this case the rigid faces have exactly one common edge. Any two of four points can be taken as the endpoints of this edge., i.e. there are 6 ways to choose this common edge. Further, if a common edge is chosen there are two ways to construct other two edges with endpoints among

the original four points: one endpoint of the common edge can be connected to one of the remaining two points to obtain one more edge and then the third edge is uniquely determined by connecting the other endpoint of the common edge with the other remaining point. So, all together we have $6 \times 2 = 12$ ways to construct parallelepipeds in the considered case.

3. **Case** $k = 1$**.** In this case there is exactly one rigid face and among 3 chosen vertices of it there is exactly one that belongs to two edges. Call it A. Then the fourth chosen point must be the opposite vertex of the parallelepiped.Call it B. The parallelepiped is uniquely determined by the choice of A and B. Any of four given points can be taken as A, which gives 4 ways and, if A is chosen, any of the remaining 3 points can be taken as B. So, all together we have $4 \times 3 = 12$ ways to construct parallelepipeds in the considered case.

4. **Case** $k = 0$**.** In this case there are no rigid faces, every face contains exactly two chosen vertices, and they are not in the same edge. So any choice of two disjoint pairs of points among the given four points defines two skew lines, which are diagonals of two parallel faces of the parallelogram and so these two faces can be uniquely reconstructed from this two skew lines. Since we have three choices of two disjoint pairs of points among the given four points, in this way we can reconstruct the unique parallelepiped from four points in the considered case.

Combining all four cases we have $4 + 12 + 12 + 1 = \boxed{29}$ parallelepipeds.