Math Contest BC Exam Solution
November 4, 2023

Directions: If units are involved, include them in your answer.

1. Suppose the hour hand and minute hand of a clock make an angle of $10°$. Assuming the hours and minutes are integers, determine the first time from midnight to noon that this occurs. Write your answer in the form $h : m$, where h represents hours and m represents minutes.

2. Consider the collection of all points, obtained by the reflection of point $B(0, 1)$ across every line passing through point $A(1, 0)$. What is the length of the curve formed by connecting these reflected points?

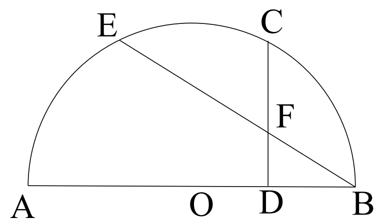
3. Given rectangle $ABCD$, points P and Q lie on sides $\overline{AB}$ and $\overline{CD}$ respectively. The line segments $\overline{PC}$, $\overline{PD}$, $\overline{QA}$, and $\overline{QB}$ collectively form six triangles and one quadrilateral. Determine the area of this quadrilateral when the areas of the two triangles containing sides $\overline{AD}$ and $\overline{BC}$ are given as 20 and 23 respectively.

4. Suppose the sum of the lengths of all edges of a rectangular prism (or a cuboid) is 64, and the length of a diagonal is $7\sqrt{2}$. Find the surface area of the rectangular prism.

5. How many natural numbers less than or equal to 1000 have exactly 3 factors?

6. What is the value of $\sqrt{11 \cdot 12 \cdot 13 \cdot 14 + 1}$?

7. Suppose C and E are on the semicircle with diameter $AB = 3$. Let D be a point on the segment $\overline{AB}$ such that $\overline{CD} \perp \overline{AB}$ and F is the point of intersection of $\overline{EB}$ and $\overline{CD}$ as in the figure. Find $\dfrac{BE}{BD}$ if $BF = 1$.



8. Find $x^6 + y^6$ if $x + y = 1$ and $x^3 + y^3 = 16$.

9. Consider a sequence of numbers 1000^2, 1001^2, 1003^2, ... Erase the two last digits from each of these numbers. How many first terms in the resulting sequence form an arithmetic progression?

10. How many three-digit numbers satisfy the following property: two of their digits are equal, and the third one differs from these by 1?

11. Suppose that increasing the bus fare by $x\%$ results in a $\frac{x}{2}\%$ decrease in the number of passengers. To achieve an 8% increase in revenue, what percentage increase in the fare should be implemented? Find the required fare increase percentage assuming that the fare increase does not exceed 50%.

12. What is the value of the natural number n for which the number of factors of $2^n(3^n+3^{n+1})$ is 99?

13. In $BC + EXAM = 2023$, all letters correspond to different digits, $B \neq 0$, $E \neq 0$. Among all solutions, find the maximal possible value of $EXAM$.

14. Consider a function $f : X \to X$ for the set X of non-negative integers. Find $f(2023)$ if $f(f(n)) + 2f(n) = 3n + 4$.

15. If a positive integer n can be represented as three-digit numbers, abc in base 6 and cab in base 9, what is the decimal representation of n?

16. There are 15 1's arranged in a row, and you can insert either a plus (+) or a minus (-) sign between every two consecutive 1's. How many different ways can you do this such that the result of the calculation equals 7?

17. Suppose E is a point inside the square $ABCD$ with $AE = 1$, $DE = 2$, and $CE = 3$. Find the diagonal AC.

18. Let A and B be two objects initially positioned at opposite points along a straight line. When they both travel at their original constant speeds, it takes 30 minutes for them to meet each other. If A doubles their speed while B maintains the original speed, they meet in 25 minutes. Determine the time in minutes it will take for them to meet if B doubles their speed while A retains the original speed.

19. Given triangle $\triangle ABC$, suppose M and N are trisection points of $\overline{BC}$, and $\overline{BE}$ is a median. Line segments $\overline{AM}$ and $\overline{AN}$ divide $\overline{BE}$ into three parts with ratios $a : b : c$. Find the ratio $a : b : c$.

20. Suppose $\angle EBD = \angle EDA = \angle DAC$ for points E and D in $\triangle ABC$. Let m_1, m_2, and m_3 be perimeters of $\triangle ABC$, $\triangle EBD$, and $\triangle ADC$. Find the maximum of $\dfrac{m_2 + m_3}{m_1}$.