CD Exam Solutions

TAMU High School Contest

11/04/2023

1. In $CD + EXAM = 2023$, all letters correspond to different digits, $C \neq 0$, $E \neq 0$. Among all solutions, find the maximal possible value of $EXAM$.

Solution: Since $EXAM < 2023$ and $E \neq 0$, we have $E = 2$ or $E = 1$.

If $E = 2$, then $X = 0$, $CD + AM = 23$. Note that neither of C, A can be 0 or 2, hence $A = C = 1$ which is impossible. Thus $E = 1$.

The largest number that starts with 1 and consists of four distinct digits is 1987. If $EXAM = 1987$, then $CD = 36$.

2. Each year, the sales went up by $x\%$ in comparison with the previous year. In 2020, the sales were more than \$100000, and in 2022, the sales were 44% more than in 2020. Find x.

Solution: Let A be the sales in 2020. Then the sales in 2021 were $\left(1 + \frac{x}{100}\right) A$, and the sales in 2022 were $\left(1 + \frac{x}{100}\right)^2 A = 1.44A$.

Since $A > 0$, we have $\left(1 + \frac{x}{100}\right) = \sqrt{1.44} = 1.2$, hence $x = 20$.

3. Out of students who attend a chess club, more than 40% but less than 50% also play tennis. What is the minimal possible number of students in the chess club?

Solution: The most straightforward way to solve this problem is to compute 40% and 50% of the first several numbers until there is an integer number between them.

n	$0.4n$	$0.5n$
1	0.4	0.5
2	0.8	1
3	1.2	1.5
4	1.6	2
5	2	2.5
6	2.4	3
7	2.8	3.5

4. S_1 is the sum of even numbers between 1 and 99, and S_2 is the sum of odd numbers between 100 and 200. Find $S_2 - S_1$.

Solution: Note that there are 49 even numbers between 1 and 99 and there are 50 odd numbers between 100 and 200. Thus

$$
\begin{aligned}
S_1 &= 2 + \cdots + 98 = \frac{(2+98)\times 49}{2} = 49 \times 50,\\
S_2 &= 101 + \cdots + 199 = \frac{(101+199)\times 50}{2} = 150 \times 50,\\
S_2 - S_1 &= (150 - 49) \times 50 = 5050.
\end{aligned}
$$

Solution:

If we add 0 to the list of even numbers, then the answer stays the same, but we can group all of the numbers in pairs:

$$
\begin{aligned}
S_2 - S_1 &= (101 + \cdots + 199) - (0 + 2 + \cdots + 98)\\
&= (101 - 0) + (103 - 2) + \cdots + (199 - 98) = 101 \times 50 = 5050.
\end{aligned}
$$

5. Compute:

$$\left(1 - \frac{1}{101^2}\right)\left(1 - \frac{1}{102^2}\right)\cdots\left(1 - \frac{1}{200^2}\right).$$

Solution: We can rewrite each term $1 - \frac{1}{n^2}$, $n = 101, \ldots, 200$, as

$$1 - \frac{1}{n^2} = \frac{n^2 - 1}{n^2} = \frac{(n-1)(n+1)}{n^2}.$$

If we do that with each term, then most numbers in the numerator and in the denominator will cancel: each n^2 in the denominator cancels with $n = (n-1)+1$ in the numerator of the previous fraction and $n = (n+1) - 1$ in the numerator of the next fraction. Once we cancel all these terms, we get $\frac{100\times 201}{101 \times 200} = \frac{201}{202}$.

6. Alice has six sticks with lengths 4, 5, 6, 12, 13, 14. She wants to choose three sticks and make a triangle out of them. How many different triangles she can get?

Solution: Let us say that the sticks of lengths 4, 5, and 6 are "short", and the other three are "long".

If Alice takes two short sticks and one long stick, then the triangle inequality will not hold, hence it will not be possible to make a triangle out of these sticks.

If Alice takes three short sticks, then the traingle inequality holds: $6 < 4 + 5$. The same is true if she takes at least two long sticks ($14 < 12 + 4$). So, in all these cases it will be possible to make a triangle out of these sticks.

So, there are $3 \times 3 = 9$ "bad" triples and $1 + 1 + 3 \times 3 = 11$ "good" triples.

7. Let ABC be an equilateral triangle with side 1. Let $ABB'A''$, $BCC'B''$, and $CAA'C''$ be squares constructed on the sides of ABC outside of ABC. Find the area of the hexagon $A'A''B'B''C'C''$.

Solution: Each triangle ABC, $AA'A''$, $BB'B''$, and $CC'C''$ has area

$$\frac{1\cdot 1}{2}\sin(60^\circ) = \frac{1\cdot 1}{2}\sin(120^\circ) = \frac{\sqrt{3}}{4}.$$

Each square has area 1, hence the total area is $3+\sqrt{3}$.

8. How many three-digit numbers satisfy the following property: two of their digits are equal, and the third one differs from these by 1?

Solution: If the two digits are equal to 0 and the third one is 1, there is one such number. If the two digits are $1,\dots,8$ and the third one is bigger by 1, we have $8\times 3 = 24$ such numbers. If two digits are equal to $2,\dots,9$ and the third one is smaller, we have $8\times 3 = 24$ such numbers. Finally, if two digits are 1 and the third one is zero, this is 101 or 110. So, all together we have $2\times 24+1+2 = \mathbf{51}$ numbers

9. Find the sum of all digits of the number $4^7\cdot 5^{10}$.

Solution: Note that $4^7\cdot 5^{10} = 2^{14}\cdot 5^{10} = 2^4\cdot 10^{10}$, so the number is 16 with ten zeros. The sum of all digits equals 7.

10. The average age of three siblings Alice, Bob, and Charlie is x. It will become $2x$ when Charlie gets as old as Bob is now. It will become $1.5x$ when Bob gets as old as Alice is now. It will become 30 when Charlie gets as old as Alice is now. Find Bob's age.

Solution: Let a, b, c be the ages of the siblings. The average age will increase by $b-c$ in the first case and by $a-b$ in the second case. Thus $b-c=x$ and $a-b=0.5x$. When Charlie gets as old as Alice is now, the average age will increase by $a-c=1.5x$ and become $2.5x=30$. Thus $x=12$. The three ages are c, $c+12$, $c+18$, with the average $12=c+10$, thus $c=2$. Siblings are 2, 14, and 20 years old.

11. A configuration of planes and points contains 12 planes and 32 points. Each plane contains 10 points. Some points are contained in 3 planes from the configuration and the other points are contained in 5 planes. How many points are contained in 3 planes?

Solution: Let x be the number of points that are contained in 3 planes. Then $32-x$ points are contained in 5 planes. Hence the number of pairs of a point and a plane that contains this point equals $3x+5(32-x) = 160-2x$. On the other hand, each plane contains 10 points, hence this number is equal to 120. Therefore, $160-2x=120$, $x=20$.

The configuration described in the problem exists, e.g., the vertices and faces of the great dodecahedron.

12. In the sequence of 2023 nonzero numbers, every number (except the first and the last one) is the product of its left and right neighbors. The fourth number is 5. Find the product of all numbers.

Solution: Consider six consequtive numbers in the sequence. If the first two of them are x, y, then these numbers are $x, y, y/x, 1/x, 1/y, x/y$. Thus the product of any 6 consequtive numbers is 1. Since $2023\equiv 1 \pmod 6$, the product of all 2023 numbers equals the first number x, and since the fourth number is $1/x$, we have $x=1/5$ and the product is $1/5$.

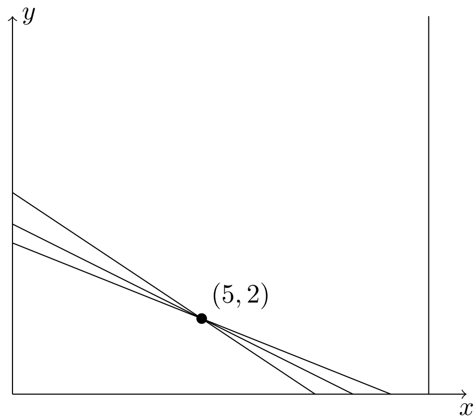
13. In the right triangle ABC with right angle B the legs have lengths $AB = 6$ and $BC = 8$. A circle passes through B and contains midpoints D, E, F of the sides AB, BC, AC. This circle intersects AC on one more point G, different from F. Find AG.

Solution: Due to the intersecting secants theorem, $AG \cdot AF = AD \cdot AB$ (this also follows from the fact that the triangles ABF and AGD are similar). We know $AB = 6$, $AD = \frac{AB}{2} = 3$, $AF = \frac{AC}{2} = \frac{\sqrt{6^2+8^2}}{2} = 5$. Thus $AG = \frac{6 \cdot 3}{5} = 3.6$.

Solution: Since $BEFD$ is a rectangle, BF is the diameter of the circle. Thus $\angle BGF = 90^\circ$, hence the triangles ABG and ACB are similar, therefore $AG = \frac{AB^2}{AC} = 3.6$.

14. For positive x and y, among the inequalities $x + 2y \geq 9$, $2x + 3y \leq 16$, $2x + 5y \leq 20$, and $x \geq 11$, three are correct and one is wrong. Find the sum of all possible values of x.

Solution: The lines $x + 2y = 9$, $2x + 3y = 16$, $2x + 5y = 20$, and $x = 11$ are shown in the following figure.



If the last inequality $x \geq 11$ is true, then both $2x + 3y \leq 16$ and $2x + 5y \leq 20$ are false. Thus we have $x + 2y \geq 9$, $2x + 3y \leq 16$, and $2x + 5y \leq 20$.

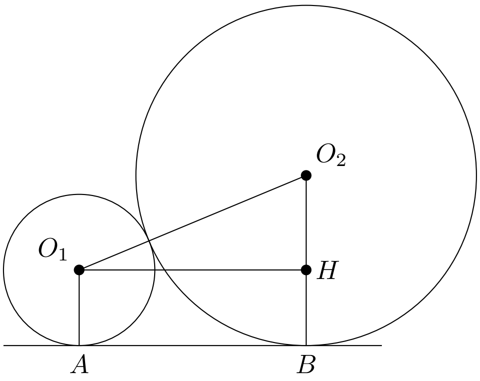
Visually, it is clear that $(5, 2)$ is the only point that satisfies these inequalities. Let us confirm this algebraically.

Adding the "$\leq$" inequalities, we get $4x + 8y \leq 36$, hence $x + 2y \leq 9$. Together with the first inequality, this implies $x = 9 - 2y$. Substituting this into the other two inequalities, we get $2 \leq y$ and $y \leq 2$, respectively. Thus $y = 2$, $x = 5$.

15. Three circles are tangent to the coordinate axis Ox at the points 0, 1/3, and 1. Also, the three circles are tangent to each other. Find the raduis of the circle that is tangent to the coordinate axis at 1/3.

Solution: If two circles ω_1 and ω_2 of radii R_1 and R_2 are tangent to each other and they are tangent to the same line at points A and B, then $AB = 2\sqrt{R_1 R_2}$.

To prove this fact, draw the perpendicular $\overline{O_1 H}$ to the line $\overline{O_2 B}$, where O_i is the center of ω_i, $i = 1, 2$. Then $O_1 H O_2$ is a right triangle with legs $O_1 H = AB$ and $O_2 H = |R_2 - R_1|$ and a hypothenuse $O_1 O_2 = R_1 + R_2$, hence by the Pythagorean Theorem we have $AB^2 = (R_1+R_2)^2 - (R_2 - R_1)^2 = 4 R_1 R_2$, thus $AB = 2\sqrt{R_1 R_2}$.



Applying this fact to each pair of circles, we get $1 = 2\sqrt{r_1r_2}$, $1/3 = 2\sqrt{r_1r_3}$, and $2/3 = 2\sqrt{r_2r_3}$, thus $r_3 = \frac{\sqrt{r_1r_3}\sqrt{r_2r_3}}{\sqrt{r_1r_2}} = 1/9$.

16. The UPS truck had eight boxes with weights of 6, 11, 13, 14, 16, 20, 27, and 30 lbs. Ann and her neighbor Ben both received deliveries. The combined weight of Ann's delivery was three times greater than the combined weight of Ben's delivery. There was only one box left in the UPS truck. Which box was it?

Solution: The total weigh of both deliveries $x + 3x$ is divisible by 4. Since the total weigh of the boxes $6 + 11 + 13 + 14 + 16 + 20 + 27 + 30$ has remainder 1 when divided by 4, the weigh of the last box should have a remainder 1. There is only one such box, 13 lbs.

The situation described in the problem is possible, if Ben got boxes weighing 11 and 20 lbs, and Ann got all other boxes but the 13 lbs box.

17. What is the sum of all possible values of c such that there exists y for which the equation $|x^2+c|x|+1| = y$ has exactly 5 solutions in x.

Solution: Since the solutions go in pairs x and $-x$, $x = 0$ must be a solution, so $y = 1$. Also, the equation $x^2 + cx + 1 = \pm 1$ must have exactly 2 positive roots.

Consider two cases.

$c \geq 0$ In this case $x^2 + cx + 1 = -1$ is impossible for positive x, and since $x^2 + cx + 1 = 1$ cannot have two positive roots, this case is impossible.

$c < 0$ In this case $x^2 + cx + 1 = 1$ has exactly one positive root $x = -c$. Thus $x^2 + cx + 1 = -1$ must have exactly one positive root. Since $c < 0$, this equation has no nonpositive roots, hence it must have a repeated root, so its discriminant equals zero. Therefore, $c^2 = 8$. Finally, from $c^2 = 8$ and $c < 0$ we get $c = -2\sqrt{2}$.

18. How many pairs (x, y) of positive integers exist such that $\sqrt{x} + \sqrt{y} = \sqrt{2023}$?

Solution: Since $2023 = 7 \cdot 17^2$, the equation is equivalent to $\sqrt{7x} + \sqrt{7y} = 7 \cdot 17$. Suppose (x, y) is a solution, where x and y are positive integers. Then $7y = (7 \cdot 17 - \sqrt{7x})^2$, which implies that $2 \cdot 7 \cdot 17 \cdot \sqrt{7x} = (7 \cdot 17)^2 + 7x - 7y$. We obtain that $\sqrt{7x}$ is a rational number. Since its square $7x$ is an integer, it follows that $\sqrt{7x}$ is, in fact, an integer. Similarly, we obtain that $\sqrt{7y}$ is an integer as well. Let $a = \sqrt{7x}, b = \sqrt{7y}$. Since the integers $a^2 = 7x$ and $b^2 = 7y$ are divisible by 7, so are the integers a and b. Let $a_1 = a/7$ and $b_1 = b/7$. Then $x = 7a_1^2$, $y = 7b_1^2$, and the equation becomes $a_1\sqrt{7} + b_1\sqrt{7} = 17\sqrt{7}$, i.e., $a_1 + b_1 = 17$. There are 16 pairs of positive integers (a_1, b_1) such that $a_1 + b_1 = 17$, so there are 16 pairs of positive integers (x, y) such that $\sqrt{x} + \sqrt{y} = \sqrt{2023}$.

19. The roots of the two polynomials $x^2 - ax + b$ and $x^2 - cx + d$ are four different positive integers. The coefficients a, b, c, d up to a permutation are four consequtive positive integers n, $n+1$, $n+2$, $n+3$. Find the product of all possible values of n.

Solution: Let $x_1 < x_2$ be the roots of the first polynomial. Then $x_1 + x_2 = a$, $x_1x_2 = b$, thus $(x_1 - 1)(x_2 - 1) = b - a + 1$. Since a and b are two of n, $n+1$, $n+2$, $n+3$, we have $-2 \le b - a + 1 \le 4$. Since x_1 and x_2 are positive integers, we have $(x_1 - 1)(x_2 - 1) \ge 0$, thus $0 \le b - a + 1 \le 4$. If $b - a + 1 = 0$, then $x_1 = 1$, hence $x_2 = a - x_1 = a - 1$, $b = x_1x_2 = a - 1$. In all other cases, $x_1 = 2$, $x_2 = a - 2$, $b = 2a - 4$. Also, $b - a + 1 = 1$ is impossible because otherwise $x_1 = x_2 = 2$.

Here are the only possible cases for x_1, x_2, a, b.

$b - a + 1$	x_1	x_2	a	b
0	1	$a - 1$	a	$a - 1$
2	2	3	5	6
3	2	4	6	8
4	2	5	7	10

The same is true for the roots x_3, x_4 and the coefficients c, d of the second polynomial. Now we need to pick two rows from this table so that the coefficients form four consequtive numbers and all the roots are distinct. Note that all equations have root 1 or 2. Without loss of generality, $x^2 - ax + b$ has root 1 and $x^2 - cx + d$ has root 2.

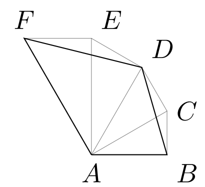
Since $b = a - 1$, $(c, d) \ne (6, 8)$. If $c = 5$, $d = 6$, then either $a = 4$ or $a = 8$; if $c = 7$, $d = 10$, then $a = 9$, $b = 8$. Here are the corresponding values of n, polynomials, and their roots.

n	$x^2 - ax + b$	$x^2 - cx + d$	x_1	x_2	x_3	x_4
4	$x^2 - 4x + 3$	$x^2 - 5x + 6$	1	3	2	3
5	$x^2 - 8x + 7$	$x^2 - 5x + 6$	1	7	2	3
7	$x^2 - 9x + 8$	$x^2 - 7x + 10$	1	8	2	5

In the case $n = 4$, two roots are equal to each other, so this case is impossible. In the remaining two cases, $n = 5$ and $n = 7$, so the product of possible values of n is 35.

20. Right triangles ABC, ACD, ADE, and AEF have right angles B, C, D, and E, respectively. These triangles are located outside of each other and $\angle BAC = \angle CAD = \angle DAE = \angle EAF = 30°$. Find the angle BDF.

Solution:



Note that rotation by 60° followed by dilation with ratio $\frac{AD}{AB} = \frac{AF}{AD} = \frac{1}{\cos^2 30°}$ maps $\overline{BD}$ to $\overline{DF}$, hence the angle between these lines is equal to the angle of rotation and $\angle BDF = 120°$.

Solution: Note that the triangles ABD and ADF are similar, because $\angle BAD = \angle DAF = 60°$ and $\frac{AB}{AD} = \frac{AD}{AF} = \cos^2 30°$. Thus $\angle BDF = \angle BDA + \angle ADF = \angle BDA + \angle ABD = 180° - \angle BAD = 120°$.