Getting Close to Sets

We will use **R** to denote the set of real numbers and **N** to denote the set of natural numbers. Sometimes real numbers are called scalars. Whenever A is a set, we will write $x \in A$ to say "x is an element of A." If A and B are sets we will write $A \subset B$ to say "A is a subset of B." If A and B are sets then we define

$$A \backslash B = \{x \mid x \in A \text{ and } x \notin B\}.$$

We say that a set A is finite if there is a natural number n and elements x_i in A for every natural number $i \leq n$ so that

$$A = \{x_i \mid i \in \mathbf{N} \text{ and } i \leq n\}$$

For example, the set $A = \{-1, 2, 5, 7, 234\}$ is a finite set, but **N** is not a finite set.

Definition 1 *Let A and B be subsets of $\mathbf{R}$ and suppose $\varepsilon > 0$. We say that A is within ε of B if and only if there exist $n \in \mathbf{N}$ and $\{x_1, ..., x_n\} \subset A$ so that for every number $x \in A \backslash \{x_1, ..., x_n\}$ there is a number $u \in B$ (dependent upon x) so that $|x - u| \leq \varepsilon$. We use the notation $(A, B) \leq \varepsilon$ to denote the statement "A is within ε of B". In addition, we write $(A, B) = 0$ whenever $(A, B) \leq \varepsilon$ for every $\varepsilon > 0$.*

Definition 2 $R^2 = \{(x, y) \mid x, y \in \mathbf{R}\}$. *The elements of $\mathbf{R}^2$ are called points.*

Definition 3 *Let $(a, b), (c, d) \in \mathbf{R}^2$ and $\alpha \in \mathbf{R}$. Then*

$$(a, b) + (c, d) = (a + c, b + d) \qquad \text{(addition of points)}$$

$$(a, b) - (c, d) = (a - c, b - d) \qquad \text{(subtraction of points)}$$

$$\alpha (c, d) = (\alpha c, \alpha d) \qquad \text{(product of scalars and points)}$$

$$|(a, b)| = \sqrt{a^2 + b^2} \qquad \text{(magnitude of a point)}$$

Definition 4 *Let A and B be subsets of $\mathbf{R}^2$ and suppose $\varepsilon > 0$. We say that A is within ε of B if and only if there exist $n \in \mathbf{N}$ and $\{(x_1, y_1), ..., (x_n, y_n)\} \subset A$ so that for every point $(x, y) \in A \backslash \{(x_1, y_1), ..., (x_n, y_n)\}$ there is a point $(u, v) \in B$ (dependent upon (x, y)) so that $|(x - u, y - v)| \leq \varepsilon$. We use the notation $(A, B) \leq \varepsilon$ to denote the statement "A is within ε of B". In addition, we write $(A, B) = 0$ whenever $(A, B) \leq \varepsilon$ for every $\varepsilon > 0$.*

1. Suppose $A = \left\{x \,\middle|\, 0 \leq x \leq \frac{1}{2}\right\}$ and $B = \left\{u \,\middle|\, u = \frac{1}{2} \text{ or } \frac{3}{4} < u \leq \frac{7}{8} \text{ or } u = 2\right\}$.

(a) Show that $(A, B) \leq \frac{1}{2}$.

(b) Is there a smallest positive number ε so that $(B, A) \leq \varepsilon$? Prove your claim.

2. Suppose $A = \left\{\frac{1}{n} \mid n \in \mathbf{N}\right\}$ and $B = \{0\}$. Show that $(A, B) = 0$ and $(B, A) = 0$.

3. Show that if A is a nonempty finite subset of $\mathbf{R}$ and B is any subset of R then $(A, B) = 0$

4. Give examples of subsets A and B of $\mathbf{R}$ to show that $(A, B) = 0$ does not necessarily imply $(B, A) = 0$.

5. Let $A, B, C \subset \mathbf{R}$ and suppose $\varepsilon > 0$ such that $(A, B) \leq \varepsilon$ and $(C, B) \leq \varepsilon$. Let $A \cup C$ denote the union of A and C, and let $A \cap C$ denote the intersection of A and C. That is

$$A \cup C = \{x \mid x \in A \text{ or } x \in C\}$$

and

$$A \cap C = \{x \mid x \in A \text{ and } x \in C\}$$

(a) Show $(A \cup C, B) \leq \varepsilon$.

(b) Show $(A \cap C, B) \leq \varepsilon$ provided $A \cap C$ is not the empty set.

6. Give examples of subsets A, B and C of $\mathbf{R}$ to show that $(A, B) = 0$ and $(C, B) = 0$ does not necessarily imply $(A, C) = 0$.

7. Give examples of subsets A, B and C of $\mathbf{R}$ to show that $(A, B) = 0$ and $(B, C) = 0$ does not necessarily imply $(A, C) = 0$.

8. Let $\mathbf{Q}$ denote the set of rational numbers and $\mathbf{I}$ denote the set of irrational numbers. It is a well know property of the real numbers that a rational number lies between any two distinct real numbers. Show that $\sqrt{2} \in \mathbf{I}$. Then use the property above to prove $(\mathbf{Q}, \mathbf{I}) = 0$ and $(\mathbf{I}, \mathbf{Q}) = 0$.

9. If $C \subset \mathbf{R}$ we define

$$C^2 = \left\{x^2 \mid x \in C\right\}$$

Show that if $A, B \subset \mathbf{R}$ and $(A, B) = 0$ then $\left(A^2, B^2\right) = 0$.

10. Let $A, B \subset \mathbf{R}$ and suppose $\alpha \in \mathbf{R}$. Define

$$A + B = \{x + y \mid x \in A \text{ and } y \in B\}$$

and

$$\alpha A = \{\alpha x \mid x \in A\}$$

(a) Show that if A and B satisfy $(A, B) = 0$ then $(\alpha A, \alpha B) = 0$.

(b) Show by example that if $A, B, C, D \subset \mathbf{R}$ satisfy $(A, C) = 0$ and $(B, D) = 0$ then it is not necessarily the case that

$$(A + B, C + D) = 0$$

11. Let $\theta > 0$ and define

$$A_\theta = \left\{ \left(\frac{n}{n+1} \cos(n\pi\theta), \frac{2n}{2n+1} \sin(n\pi\theta) \right) \mid n \in \mathbf{N} \right\}$$

and

$$B = \left\{ (x, y) \;\middle|\; x^2 + y^2 = 1 \right\}$$

(**Note:** The argument $n\theta$ above is given in radians, not degrees!!)

(a) Show that if $\theta > 0$ then $(A_\theta, B) = 0$.

(b) Suppose $\theta > 0$ is a rational number. Is there a smallest positive number ε so that $(B, A_\theta) \leq \varepsilon$. Prove your claim.

(c) Show that if $\theta > 0$ is an irrational number then $(B, A_\theta) = 0$.

(d) Show that if $\theta, \omega > 0$ are irrational numbers then $(A_\theta, A_\omega) = 0$.

12. Let $A = \left\{ \left(x, x^2\right) \mid -4 \leq x \leq 4 \right\}$ and $B = \{(x, 2x+1) \mid -4 \leq x \leq 4\}$. Is there a smallest possible number $\varepsilon > 0$ so that

$$(A, B) \leq \varepsilon?$$

Prove your claim.

13. Define

$$S = \left\{ (0,0), (1,0), \left(\frac{1}{2}, \frac{\sqrt{3}}{2} \right) \right\}.$$

Let P and J be points in $\mathbf{R}^2$. We define

$$A_1 = \{P\}$$

Then, inductively, for $n = 2, 3, ...$ define

$$A_n = \left\{ \frac{1}{2}U + \frac{1}{2}V \mid U \in A_{n-1} \text{ and } V \in S \right\}.$$

Similarly, we define

$$B_1 = \{J\}$$

Then, inductively, for $n = 2, 3, ...$ define

$$B_n = \left\{ \frac{1}{2}U + \frac{1}{2}V \mid U \in B_{n-1} \text{ and } V \in S \right\}.$$

Let $A = A_1 \cup A_2 \cup A_3 \cup \cdots$ and $B = B_1 \cup B_2 \cup B_3 \cup \cdots$. Show that $(A, B) = 0$. **Notes:** $\frac{1}{2}U + \frac{1}{2}V$ is the midpoint of the line segment joining U and V. The choice $P = (0, 0)$ leads to the set A shown in Figure 1.

14. Define $R_1 = (0, 0)$, $R_2 = (1, 0)$ and $R_3 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. For each $n \in \mathbf{N}$ let a_n be chosen from the set $\{1, 2, 3\}$, and form the number a given by the decimal expansion

$$a = 0.a_1 a_2 a_3 a_4 \cdots$$

Now create a set A as follows. Let P_0 be a point in $\mathbf{R}^2$. Set

$$P_1 = \frac{1}{2}P_0 + \frac{1}{2}R_{a_1}$$

and inductively, for $n = 2, 3, 4, ...$ define

$$P_n = \frac{1}{2}P_{n-1} + \frac{1}{2}R_{a_n}.$$

Then set $A = \{P_0, P_1, P_2, P_3 ...\}$.

(a) Show that if a is a rational number then there exist a finite number of points $Q_1, Q_2, ..., Q_m$ in $\mathbf{R}^2$ so that

$$(A, \{Q_1, Q_2, ..., Q_m\}) = 0$$

(b) Is the converse of the statement above true?

(c) Let B be a set constructed in the manner described in problem 13. Is it possible for the values a_n to be chosen in such a way that $(A, B) = 0$?