The Problems With Apportionment

How are the number of congressional representatives for each state determined?

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What does the constitution say?

In Article 1, Section 2 it states that

Representatives and direct Taxes shall be apportioned among the several States which may be included within this Union, ***according to their respective Numbers***

In Section 2 it states that

The Number of Representatives shall not exceed one for every thirty thousand, but ***each State shall have at least one Representative****;*

An Apportionment Problem

A manager at a busy restaurant needs to schedule 15 food servers to 3 different shifts.

She decides to apportion the servers according to the typical number of customers during each shift.

Breakfast averages 253 customers

Lunch averages 182 customers

Dinner averages 85 customers

Let's do the math…

253 + 182 + 85 = 520 customers

$s = 520/15 = 34.67$ customers/server

This value is known as the *standard divisor*

Here the standard divisor is the ideal number of customers per server.

In the United States in 2010

$s = 309{,}183{,}463 / 435 = 710{,}766.58$ people/representative

So how many servers per shift?

At breakfast,

253 customers/34.67 customers per server

= 7.30 servers

At lunch,

182 customers/34.67 customers per server

= 5.25 servers

At dinner,

85 customers/34.67 customers per server

= 2.45 servers

These values are known as the *quotas*

What do we do with the fractions?

This is the same problem our founding fathers faced when apportioning for the house of representatives.

Alexander Hamilton, Thomas Jefferson, and John Quincy Adams all had ideas on how to apportion for the 1790 census.

How does Hamilton's method work?

Hamilton's Method

Begin by finding the quotas. Next, round each quota down.

Shift	Quota	Number
Breakfast	253/34.67 = 7.30	7
Lunch	182/34.67 = 5.25	5
Dinner	85/34.67 = 2.45	2 +1 = 3
Total		14 →15

We have only 14 servers assigned. *Assign the remaining server to the quota that has the largest fractional part.*

How did the apportionment end up?

Remember, the standard quota was 34.67

Breakfast: 253 /7 = 36.14 customers/server
Lunch: 182/5 = 36.4 customers/server
Dinner: 85/3 = 28.33 customers/server

Hamilton's method was approved by congress but was vetoed by George Washington.
This was the first veto in the history of the United States!

Jefferson's Method

Jefferson's method was approved by congress and was used to apportion from 1790 until 1840.

In Jefferson's method the divisor is adjusted to produce the correct total number of representatives needed.

Like Hamilton's method, it begins with the quotas.

Adjusting the divisor

Begin by rounding each quota down,

Breakfast	253/34.67 = 7.30	7	253/(7+1) = 31.63	8
Lunch	182/34.67 = 5.25	5	182/(5+1) = 30.33	5
Dinner	85/34.67 = 2.45	2	85/(2+1) = 28.33	2
Total		14		15

We have only 14 servers assigned so we check to see what the divisor would for be each group, if they received the extra server. The divisor closest to the standard divisor (34.67) gets the extra server.

Jefferson's method was used from 1790 until 1840

However, Jefferson's method strongly favors large states.

In 1842, Daniel Webster's method was adopted.

In his method, the quotas are rounded to the nearest number.

Then if the total number of representatives is not correct, the divisors are adjusted to get the correct total.

Webster's Method

Round each quota to the nearest integer ,

Breakfast	253/34.67 = 7.30	7	253/7.5 = 33.73	7
Lunch	182/34.67 = 5.25	5	182/5.5 = 33.09	5
Dinner	85/34.67 = 2.45	2	85/2.5 = 34	3
Total		14		15

We have only 14 servers assigned so we check to see what the divisor would for be each group by adding ½ to each denominator. The divisor closest to the standard divisor (34.67) gets the extra server.

In 1852 Hamilton's method was adopted

But due to politics, some odd things occurred.

These include adjusting the number of representatives to make the apportionment using Webster's method and Hamilton's method come out the same.

In 1920, they simply did not reapportion – in direct violation of the constitution.

Huntington's Equal Proportions Method was adopted for the 1940 apportionment and has been used since then.

Huntington's Equal Proportions Method

This method rounds the quotas using the geometric mean.

The arithmetic mean of two numbers a and b is $(a+b)/2$

The geometric mean of two numbers a and b is $\sqrt{ab}$

When rounding a number between 1 and 2, use $\sqrt{1 \times 2} = 1.41$

So 1.42 would round up to 2.

When rounding a number between 0 and 1, use $\sqrt{0 \times 1} = 0$

So 0.22 would round up to 1.

Paradoxes and Problems

A city has 3 districts (total population 13,960) and 10 representatives. Apportion using Hamilton's method:

District	Size	Quota	N	
North	6021	6021/1396 = 4.31	4	
East	5930	5930/1396 = 4.25	4	
West	2009	2009/1396 = 1.44	1	+1

The number of representatives is increased to 11. What happens?
Alabama Paradox (discovered in 1880).

District	New Quota	N	
North	6021/1269.09 = 4.74	4	+1
East	5930/1269.09 = 4.67	4	+1
West	2009/1269.09 = 1.58	1	

Divisor methods are not perfect

A company will hire 200 new workers for its four facilities around the state.

The workers will be apportioned using Jefferson's method according to the current number of workers at each facility.

Currently there are a total of 18,834 workers.

s = 18,834/200 = 94.17

Location	current	quota	N	new divisor	
Abilene	12,520	132.95	132	12,520/133 = 93.14	+1
Beaumont	4,555	48.37	48	4,555/49 = 92.96	
Corpus C.	812	8.62	8	812/9 = 90.22	
Dallas	947	10.56	10	947/11 = 86.09	
Total	18,834		198		

Apportion again with the new divisor of d = 12,520/133 = 93.14

Our new divisor is d = 94.14. Find the quotas

Location	current	new quota	N	divisor	
Abilene	12,520	12,520/94.14 = 133	133	93.43	+1
Beaumont	4,555	4,555/94.14 = 48.39	48	92.56	
Corpus C.	812	812/94.14 = 8.63	8	90.22	
Dallas	947	947/94.14 = 10.06	10	86.09	
Total	18,834		199		

Which divisor is the closest to 94.14?

Quota Rule: *The number assigned to each unit must be the standard quota rounded up or down.*

Divisor methods can violate the quota rule.

Interested in learning more?

http://www.math.tamu.edu/~epstein/Math167/

(see the Chapter 14 material)

http://www.ams.org/samplings/feature-column/fcarc-apportion1

http://www.census.gov/history/www/reference/apportionment/methods_of_apportionment.html