NUMERICAL ANALYSIS QUALIFIER
May 23, 2006

Do all of the problems below. Make sure that you show your work in yes/no problems (simply answering yes or no will receive no credit).

Problem 1. (a) Let $A$ be an $n \times n$ matrix and $\| \cdot \|_1$ denote the norm on $\mathbb{R}^n$ given by $\|v\|_1 = \sum_{i=1}^n |v_i|$. Show that

$$\|A\|_1 = \max_{j=1}^n \sum_{i=1}^n |A_{i,j}|.$$ 

(b) Let $\| \cdot \|_2$ denote the norm on $\mathbb{R}^n$ given by $\|v\|_2 = (\sum_{i=1}^n |v_i|^2)^{1/2}$. For $A = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$, compute $\|A\|_2$.

Problem 2. The following questions relate to (multistep) ODE schemes for solving $x'(t) = f(x(t), t)$.

(a) Show that the scheme

$$x_n - x_{n-1} = h(\theta f_n + (1 - \theta)f_{n-1})$$

is $A$-stable for $1/2 \leq \theta \leq 1$.

(b) Consider the scheme

$$x_n - 3x_{n-1} + 2x_{n-2} = -\frac{1}{2}h(f_n + f_{n-1}).$$

Is it consistent? Is it stable?

(c) Consider the scheme

$$x_n - x_{n-2} = h(f_n - 3f_{n-1} + 4f_{n-2}).$$

Is it stable? Compute its order.

Problem 3. Consider the finite element space defined on triangles in $\mathbb{R}^2$ consisting of piecewise cubic functions $P^3$. Consider as degrees of freedom:

(i) :The function values at the vertices.

(ii) :The values of the first derivatives at the vertices.

(iii) :The value of the function at the barycenter.

(a) Show that the degrees of freedom above form a unisolvent set. Do this by using properties of the polynomials and not by writing down a huge system and trying to show that it is nonsingular.
(b) State a theorem which provides a criterion that you can apply to determine when an assembled finite element space is $H^1$-conforming.

(c) Show that the assembled finite element space corresponding to this problem is $H^1$-conforming.

(d) Prove or disprove: The assembled finite element space corresponding to this problem is $H^2$-conforming.

**Problem 4.** Consider the one dimensional wave equation,
\[
\zeta_{tt} - \zeta_{xx} = 0, \quad \text{for } (x, t) \in (0, 1) \times (0, T],
\]
(4.1)
\[
\zeta(0, t) = \zeta(1, t) = 0, \quad \text{for } t \in (0, 1],
\]
\[
\zeta(x, 0) = \zeta_0(x), \quad \zeta_t(x, 0) = \eta_0(x), \quad \text{for } x \in [0, 1].
\]

(a) Describe the (time continuous) semi-discrete approximation to (4.1) based on finite differences on a uniform grid in space consisting of $m$ internal nodes.

(b) The semi-discrete method of Part (a) above can be written as a system of ODEs of the form
\[
Z_{tt} + AZ = 0, \quad \text{for } t > 0
\]
\[
Z(0) = Z_0, \quad Z_t(0) = N_0.
\]

Here $Z(t), Z_0, N_0 \in \mathbb{R}^m$, $A$ is a symmetric and positive definite $m \times m$ matrix, and $Z_0$ (resp. $N_0$) interpolates $\zeta_0$ (resp. $\eta_0$). Let $N = Z_t$ then the above system leads to the first order system
\[
N_t + AZ = 0, \quad Z_t - N = 0.
\]
(4.2)

Consider the following fully discrete scheme for (4.2) with step-size $k$:
\[
\frac{Z_{n+1} - Z_n}{k} - N_n + \frac{k}{2}AZ_{n+1} = 0,
\]
(4.3)
\[
\frac{N_{n+1} - N_n}{k} + AZ_{n+1} = 0.
\]

Show that $Z_k$ satisfies a three term recurrence by eliminating $N$.

(c) Show that (4.3) is unconditionally stable by using the three term relation of Part (b) above and expanding in terms of the eigenvectors of $A$. 