Goldbach's Conjecture

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Abstract

Goldbach's Conjecture, that every number can be written as a sum of two primes, has been neither proven nor disproven since it was posited in the 1740's. The most recent research into the topic (by Oliveira e Silva in 2005) showed that every even number less than 3×10^{17} can be written as a sum of two primes. This research shows that there are large numbers of even numbers, (some possibly beyond this limit) for which we can also easily confirm as being the sum of two primes. While this is not a complete proof of Goldbach's Conjecture, it does confirm that it holds locally at a large number of n.

1 Current Research

Before we continue, there are two important definitions we need to posit.

Definition 1. *Given a number, n, we may say that any number m less than n is a totive of n iff $m \nmid n$.*

Definition 2. *$\phi_C(n)$ is the number of composite totives of n.*

Theorem 3. *$\phi_C(n) < \pi(n)$ implies that n can be written as a sum of two primes.*

Proof. In general, even numbers that can be written as the sum of two primes do not have a unique way of doing so; therefore, it is useful to define an algorythm for writing such an algorithm.

Let $\mathbb{P}$ be the set of all primes

Now define the sets,

$$\mathbb{P}_0(n) := \{p \in \mathbb{P} | p < n\} \tag{1}$$

$$\mathbb{P}_N(n) := \mathbb{P}_{N-1}(n) - \{\max[\mathbb{P}_{N-1}(n)]\} \ \forall N \geq 1 \tag{2}$$

Now find

$$m := \min[\{N \in \mathbb{N} | (n - \max[\mathbb{P}_N(n)]) \in \mathbb{P}\}] \tag{3}$$

n may then be written as

$$n = \max[\mathbb{P}_m(n)] + (n - \max[\mathbb{P}_m(n)])) \tag{4}$$

Suppose there exists a D that cannot be written as a sum of two primes. Then

$$\forall n \in \mathbb{N}\, (D - \max[\mathbb{P}_n(D)]) \notin \mathbb{P} \tag{5}$$

Then clearly for any prime $p_i < D$, $D - p_i$ is a composite number.

Another form of this assertion is that for any D that cannot be written as the sum of two primes, for every prime p_i there exists a composite c_i (both of which are less than D), such that

$$D = p_i + c_i \tag{6}$$

We may observe that $p_i + c_i = D$ implies that $p_i = D - c_i$. Now, if D and c_i shared a common factor (Say, for example, $D = \kappa\, d$ and $c_i = \kappa\, k_i$), then we could write

$$p_i = \kappa\, d - \kappa\, k_i = \kappa\, (d - k_i) \tag{7}$$

But this implies that p_i is not prime.

Therefore, c_i and N can have no common factors (Because c_i is less than N, we can say that c_i is a totive of N). Therefore, this implies that any D (even) that cannot be written as a sum of two primes must satisfy:

$$\phi_C(D) \geq \pi(D) \tag{8}$$

If D did not satisfy this, then there would not be enough possible c_i to go with every p_i. Therefore, any number for which the negation of (8) is true can be written as a sum of two primes. □

2 Future Research

Unfortunatly, this relationship does not prove Goldbach's Conjecture, it only shows that any 'disproofs' will likely be isolated cases.

Research is continuing towards the following:

Theorem 4. *There exists no* n *such that* $\phi_C(n+m) \geq \pi(n+m)$ *for all* $m \in \mathbb{N}$.

This appears to be true because we can construct large k with very low $\phi_C(k)$, but I am not certain I have obtained a proof at this time.

Proof of theorem 4 would show that there are infinitly many even numbers that can be written as a sum of two primes.

Theorem 5. *Goldbach's Conjecture*

It is not hard to see that because of the relation $D = p_i + c_i$ we are guaranteed that if D cannot be written as a sum of two primes either $p_i < \frac{D}{2}$ and $c_i > \frac{D}{2}$ or $p_i > \frac{D}{2}$ and $c_i < \frac{D}{2}$. Because primes are not distributed symmetricly about $\frac{D}{2}$ this approach (and perhaps one with slightly smaller bounds) could show that even if $\phi_C(D) \geq \pi(D)$ the totives of D will not be distributed in such a way to allow D to be written as $D = p_i + c_i$ for all p_i.

Proof of theorem 5 is the ultimate goal for the research.