

End of Program Report

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Introduction

Our research experience has been centered around the following conjecture and its corollary (here $Z_+(f)$ refers to the positive real zero set of f):

Conjecture: (Main Theorem) *Given $f \in \mathbb{R}[x_1, \dots, x_n]$ an honest $(n+2)$ -nomial, $Z_+(f)$ has topology isotopic to a quadric hypersurface of the form*

$$x_1^2 + \dots + x_j^2 - (x_{j+1}^2 + \dots + x_n^2) = \varepsilon$$

where j and the sign of ε are computable in polynomial time (for fixed n) from the support A and coefficients of f .

Corollary: *Given $f \in \mathbb{R}[x_1, \dots, x_n]$ an honest $(n+2)$ -nomial, $Z_+(f)$ has at most two connected components.*

Generally speaking, our task was to understand our goal along with some possible approaches, to research relevant information keeping in mind how it might help us reach our goal, and to keep an eye out for opportunities to generalize our expected results or simplify current results. We also had to continually make sure that we were ready to present our work to an audience.

What We've Learned

Over the course of the past eight weeks, we were lectured on and read about a wide variety of topics. First, we learned some basic tools and motivations within the field of algebraic geometry. We began with resultants and discriminants, completing assignments which required us to demonstrate a basic understanding of these tools. We later learned of the way to represent resultants by discriminants given to us by Cayley's trick [2], and also how to represent the discriminant as a resultant using the definition of the discriminant as the common root of all of the partial derivatives and the function. Discriminants are very useful to our project. More specifically the connected components of the complements of discriminant varieties (zero sets of discriminants), which are called "discriminant chambers", each give us a set of polynomials with the same topology.

The assignments also introduced us to ways of using linear algebra to simplify problems in algebraic geometry. For example, we learned how to use linear algebra to find a simple form for the discriminant of n -variate $(n+2)$ -nomials as done in [3]. We developed skills in polynomial manipulation, utilizing multiple simplification techniques such as monomial, linear, and exponential variable changes. Using these we could get our system into a nice canonical form which was much easier to work with.

We also studied Viro Diagrams, which are a combinatorial approach to determining the topology of varieties. Related to Viro diagrams is Frederic Bihan's work on hypersurfaces, which illustrates the special case of working with n -variate, $(n+2)$ -nomials. When polynomials are supported on a circuit, we have more options available to us, including results such as his Theorem 10.

Topology provided an additional selection of tools. First establishing an understanding of basic topological structures and vocabulary, we then began to search within differential and algebraic topology for useful ideas. We use Morse theory, a subset of differential topology, to prove our corollary, while algebraic topology provides homological properties that will be useful in our algorithm. We are concerned with isotopic and diffeotopic topological equivalence in the main theorem, for which we developed a grasp of homeomorphism, diffeomorphism, Betti numbers, connectedness, and manifolds, among other topological concepts and invariants.

Finally, we studied complexity theory. It was necessary to review the basic complexity classes, but it was also important that we research some of the lesser known complexity classes. Since we're working with real numbers, it was advantageous to gain an understanding of the BSS model of computation. Under the BSS model the basic operations on real numbers are treated as some which happen in constant time. This will serve to simplify the classification of our problem greatly since we no longer have to take into account the bit lengths of our numbers and how they're affected throughout the process of our algorithm.

Process and Results

We began by establishing a familiarity with the above topics through intensive reading and exercises. Then we attempted to prove the main theorem early on using what we'd learned about Morse Theory, but we were unsuccessful. We needed more of a solid understanding of our tools before we could make any progress on that proof. So we instead shifted our attention to proving the corollary using the same machinery. We hoped this would give us a better understanding while at the same time giving us a concrete result to present. This turned out to be no easy task.

Proving the Corollary

We use induction, where in our base case, we work with a univariate trinomial

$$f(x) = c_0 \pm c_1x \pm c_2x^2$$

which has at most two positive real roots by Descartes's Rule, and thus two connected components. Next we assume our inductive step: that for any $(n-1)$ variate $(n+1)$ -nomial, $Z_+(f)$ has at most two connected components. Then we consider an arbitrary n -variate, $(n+2)$ -nomial,

$$f_1(x_1, \dots, x_n) = c_0x^{a_0} + c_1x^{a_1} + \dots + c_{n+1}x^{a_{n+1}}$$

(We may assume that each $a_{ij} \geq 0$ by multiplying by the appropriate monomial, which leaves the positive zero set unaffected). We utilize the simplification methods we have learned to change our polynomial into a simplified exponential sum

$$f_3(x_1, \dots, x_n) = 1 \pm e^{x_1} \pm \dots \pm e^{x_n} \pm ke^{b \cdot x}$$

examining $Z_{\mathbb{R}}(f_3)$ instead of $Z_+(f_1)$. In doing so, we need to study the topological aspects we had been introduced to much more deeply. Specifically, we work with Morse Theory to examine the level sets of $Z_{\mathbb{R}}(f)$, treating $Z_{\mathbb{R}}(f)$ as a manifold M . Initially we compute the general form of our Hessian matrix in order to examine the possible eigenvalues—and thus potential "surgeries"—for our particular case. However, the characteristic polynomial is particularly difficult to compute. As such, we decided to use a different approach.

Our second approach involves examining the relationship between our inductive step, critical points, and path-connectedness. Here, critical points must satisfy the following system of equations:

$$\mathbf{H} = \begin{cases} \pm e^{x_1} = \pm \gamma \alpha_1 e^{\alpha x} \\ \vdots \\ \pm e^{x_{n-1}} = \pm \gamma \alpha_{n-1} e^{\alpha x} \end{cases}$$

We have two cases: first, \mathbf{H} has no solutions; second, \mathbf{H} has at least one solution. Examining our manifold M , our inductive step and Theorem 2.2 from *Differential Topology* by Morris W. Hirsch [1], we proved that, in the first case where our Morse function on M gave no critical points, we always had at most two connected components.

In the second case, where our Morse function did yield critical points, that the critical points must satisfy \mathbf{H} allows us to substitute and simplify to obtain

$$1 \pm e^{x_n} \pm \gamma' e^{\alpha_n x_n}$$

By Descartes's Rule, there can be at most two solutions, i.e., at most 2 critical points. Assuming there is one critical point, we show that there is at most one connected component. We use, again, Theorem 2.2 with our inductive step, and then the fact that locally to a critical point M is "like" a smooth quadric hypersurface [1] to write a concatenated path from $m \in M$ to the critical point, which can be done for any $m \in M$. This means that m shares the same path connected component as the critical point $\forall m \in M$, and since there is only one critical point, there must be only one path connected component.

We use the same approach for the case of 2 connected components, except instead we show that there exists a path from m to one of the two critical

points, demonstrating that m must lie in a connected component with either of the critical points. We still must show that it cannot be the case that we have at least three connected components and two critical points, using restrictions on surgery types.

Finally, we must consider the case with singularity, by showing first that if there is a singularity, then it is unique (an obvious fact given the system of equations it must solve). If the singular point's uniqueness holds, we can use stratified Morse Theory and similar reasoning as in the case of the single critical point.

This would prove both of our cases, implying that for f an honest, n -variate, $(n + 2)$ -nomial, $Z_+(f)$ has at most 2 connected components, thus proving our corollary.

Future Directions

Theorem and Paper

Understanding connectedness and number of components are key parts in understanding the topology of $Z_+(f)$. As we develop our algorithm, we will use this fact to understand $Z_+(f)$ and to gain intuition as to which quadric hypersurface a given positive real zero set may yield.

Dr. Rojas provided us with a copy of a paper-in-progress, which we read through. We will be working with Rojas as co-authors along with Dr. Bihan, who has previously co-authored work with Dr. Rojas. [4]

Moving forward, we will finish the completely formalized write-up of our corollary proof, which will be integrated into our paper with Dr. Rojas and Dr. Bihan. We also hope to revisit our previous approach to proving the main theorem with our refined understanding of the machinery involved. Of course we will try other approaches as well, including looking at modified versions of the discriminant chamber idea (which take into account only positive orthants), and also looking at level sets of discriminant varieties.

Throughout the coming semester we will continue to work on the paper and the algorithm we established in the beginning, aiming to complete formulation and implementation sometime in December. Over this time period we will continue to communicate with Dr. Rojas about the paper and assist with any revisions, edits, or additions we can, with the hopes of eventually publishing the paper. After the paper is published, we would like to try to generalize this work and/or explore other avenues related to our problem if possible.

Furthermore, we will present our work at MAA MathFest 2013 in Connecticut, as well as look to future conferences at which to present.

References

- [1] Morris W. Hirsch *Differential Topology* 1976: Springer-Verlag, N.Y.
- [2] I.M. Gelfand, M.M. Kapranov, and A.V. Zelevinsky *Discriminants, Resultants, and Multidimensional Determinants* 2008: Birkhäuser Boston, N.Y.
- [3] “Faster Real Feasibility via Circuit Discriminants,” (by Frederic Bihan, J. Maurice Rojas, and Casey Stella), proceedings of ISSAC 2009 (July 28-31, Seoul, Korea), pp. 39-46, ACM Press, 2009.
- [4] “Faster Topology Computation for Positive Zero Sets of Certain Sparse Polynomials,” F. Bihan, E. Refsland, J. R. Rennie, and J. M. Rojas, in progress.