ZEROS OF MAASS FORMS

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Abstract. We study the location of the zeros of the Maass form obtained by applying the Maass level raising operator $R_k = i\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) + \frac{k}{y}$ to E_k. We find that this Maass form has the same number of zeros on the bottom arc of $\mathcal{F}$ as E_{k+2}, and conjecture that all of its zeros in $\mathcal{F}$ lie on this arc. We note that this seems to hold for the Maass form obtained by applying the level raising operator multiple times to the Eisenstein series.

1. Introduction and Statement of Results

The zeros of modular forms have been well studied over the years. For example, in a very influential paper [5], Rankin and Swinnerton-Dyer showed that all of the zeros of the weight k Eisenstein series, E_k, in the standard fundamental domain $\mathcal{F}$ lie on the bottom arc $\mathcal{B} = \left\{e^{i\theta} : \frac{\pi}{2} \leq \theta \leq \frac{2\pi}{3}\right\}$ of $\mathcal{F}$. In stark contrast, it was shown by the work of Rudnick [6] and Holowinsky-Soundararajan [2] that the zeros of Hecke eigenforms become equidistributed with respect to the hyperbolic measure on the fundamental domain as the weight grows. In this paper we study the location of the zeros of the Maass form obtained by applying the Maass raising operator R_k to E_k. The result is no longer a modular form, but instead a weight $k+2$ Maass form.

We find that this Maass form has the same number of zeros on $\mathcal{A}$ as E_{k+2}. Define $m(k)$ by

$$m(k) = \begin{cases} \lfloor k/12 \rfloor, & \text{if } k \not\equiv 2 \bmod (12), \\ \lfloor k/12 \rfloor - 1, & \text{if } k \equiv 2 \bmod (12). \end{cases} \tag{1}$$

Theorem 1.1. *The Maass form R_kE_k has $m(k+2)$ zeros along the arc $\mathcal{A}$.*

Since R_kE_k is no longer holomorphic, we do not have a valence formula for it. Hence, this theorem does not exclude the possibility of there being other zeros inside the fundamental domain. However, as can be seen in Figure [**?**], numerical experiments with Mathematica suggest that all the zeros are on $\mathcal{A}$.

Conjecture 1.2. *All of the zeros of R_kE_k on $\mathcal{F}$ lie on the bottom arc $\mathcal{A}$.*

2. Preliminaries

A modular form of weight k is a complex-valued function f on the upper half-plane $\mathbb{H} = \{z \in \mathbb{C}, \mathrm{Im}(z) > 0\}$, satisfying the following three conditions:

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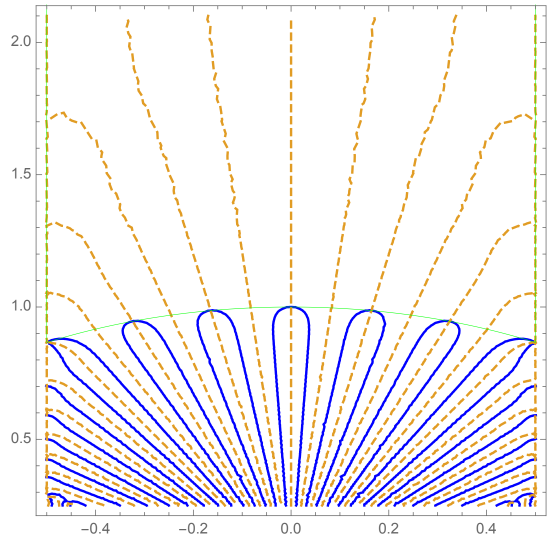

FIGURE 1. $\mathrm{Re}(R_{36}E_{36}(z)) = 0$ and $\mathrm{Im}(R_{36}E_{36}(z)) = 0$ in solid and dotted lines, respectively.

(a) For any z in $\mathbb{H}$ and any matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, f satisfies

$$f(\gamma z) = f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z). \tag{2}$$

(b) f is a complex analytic function on $\mathbb{H}$.
(c) f is required to be holomorphic as $z \to \infty$.

A Maass form is a generalisation of modular forms in which the last two conditions of modular forms are replaced by the following:

(b)′ f is an eigenvector of the Laplacian operator

$$\Delta_k = -y^2\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big) + iky\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big).$$

(c)′ f is of at most polynomial growth as $z \to \infty$.

Throughout this paper we will use the following standard notation $\Gamma = SL_2(\mathbb{Z})$, $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\rho = e^{\frac{2\pi i}{3}}$.

Let $E_k(z)$ be the Eisenstein series of weight k, k even, defined by

$$E_k(z) = \frac{1}{2}\sum_{\substack{\gcd(c,d)=1 \\ c,d\in\mathbb{Z}}}\sum (cz+d)^{-k}. \tag{3}$$

Its Fourier expansion is given by

$$E_k(z) = 1 + \frac{2}{\zeta(1-k)}\sum_{n=1}^{\infty}\sigma_{k-1}(n)e^{2\pi i n z},$$

where $\sigma_{k-1}(n)$ is the divisor function.

Let M_k denote the complex vector space of modular forms of weight k for Γ. If $f \in M_k$, the valence formula is given by

$$\frac{k}{12} = \frac{1}{2}\mathrm{ord}_i(f) + \frac{1}{3}\mathrm{ord}_\rho(f) + \mathrm{ord}_\infty(f) + \sum_{\tau \in \Gamma\backslash\mathbb{H} - \{i,\rho\}} \mathrm{ord}_\tau(f). \tag{4}$$

For details and a proof of this formula see §III.2 of [3].

Define $m(k)$ by

$$m(k) = \begin{cases} \lfloor k/12 \rfloor, & \text{if } k \not\equiv 2 \bmod (12), \\ \lfloor k/12 \rfloor - 1, & \text{if } k \equiv 2 \bmod (12), \end{cases} \tag{5}$$

and write $k = 12m(k) + s$. Note that s determines the residue class of k modulo 12. Using this and the valence formula, we obtain

$$\mathrm{ord}_i(f) \geq \begin{cases} 1, & \text{if } k \equiv 2 \bmod (4), \\ 0, & \text{if } k \equiv 0 \bmod (4), \end{cases} \tag{6}$$

and

$$\mathrm{ord}_\rho(f) \geq \begin{cases} 2, & \text{if } k \equiv 2 \bmod (6), \\ 1, & \text{if } k \equiv 4 \bmod (6), \\ 0, & \text{if } k \equiv 0 \bmod (6). \end{cases} \tag{7}$$

Define the derivative of a modular form f to be

$$Df = \frac{1}{2\pi i}\frac{df}{dz}.$$

Recall the Maass raising and lowering operators R_k and L_k on functions $f : \mathbb{H} \to \mathbb{C}$ which are defined by

$$R_k = 2i\frac{\partial}{\partial z} + \frac{k}{y} = i\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) + \frac{k}{y},$$

$$L_k = -2iy^2\frac{\partial}{\partial \bar{z}} + \frac{k}{y} = -iy^2\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right).$$

With respect to the Petersson slash operator, these satisfy the properties $R_k(f|_k\gamma) = (R_kf)|_{k+2}\gamma$ and $L_k(f|_k\gamma) = (L_kf)|_{k-2}\gamma$, for any $\gamma \in \mathrm{SL}_2(\mathbb{R})$. This is, if f transforms by (2), then R_kf and L_kf transform like modular forms of weight $k+2$ and weight $k-2$, respectively. Here R_k satisfies the relation $R_k = -4\pi D + \frac{k}{y}$. If we apply R_k to the weight k Eisenstein series, we obtain

$$\begin{aligned} R_k\left(E_k(z)\right) &= -4\pi D E_k(z) + \frac{k}{y}E_k(z) \\ &= -ki \sum_{\substack{\gcd(c,d)=1 \\ c,d \in \mathbb{Z}}} \sum c(cz+d)^{-k-1} + \frac{k}{y}\frac{1}{2}\sum_{\substack{\gcd(c,d)=1 \\ c,d \in \mathbb{Z}}} \sum (cz+d)^{-k}. \end{aligned}$$

We are interested in the properties of this form. In particular, we study the number and the location of its zeros inside the fundamental domain $\mathcal{F} = \{z \in \mathbb{H} : -1/2 \leq \mathrm{Re}(z) \leq 1/2, |z| \geq 1\}$.

3. Basic properties of zeros of $R_k(f)$.

Definition 3.1. $R_k^j = R_{k+2j-2} \circ \cdots \circ R_{k+2} \circ R_k$.

Lemma 3.1. *Suppose f satisfies* (2) *and $k \equiv 2 (\mathrm{mod} 4)$. Then $f(i) = 0$.*

Proof. Note that $(S)i = i$. Since f transforms like a weight k modular form, we have that

$$f((S)i) = (i)^k f(i). \tag{8}$$

Since i is a fourth root of unity, we have that $f(i) = 0$ whenever $k \equiv 2 \bmod (4)$. □

Corollary 3.2. *Suppose f satisfies* (2) *and $k \equiv 2 - 2j (\mathrm{mod} 4)$. Then $R_k^j f$ has a zero at i.*

Lemma 3.3. *Suppose f satisfies* (2) *and $k \equiv 2, 4 (\mathrm{mod} 6)$. Then $f(\rho) = 0$.*

Proof. Note that $(ST)\rho = \rho$. Since f transforms like a weight k modular form, we have that

$$f((ST)\rho) = (\rho + 1)^k f(\rho). \tag{9}$$

Since $\rho + 1$ is a sixth root of unity, we have that $f(\rho) = 0$ whenever $k + 2j \equiv 2, 4 \bmod (6)$. □

Corollary 3.4. *Suppose f satisfies* (2) *and $k \equiv 2 - 2j (\mathrm{mod} 6)$ or $k \equiv 4 - 2j (\mathrm{mod}\ 6)$ Then $R_k^j f$ has a zero at ρ.*

Lemma 3.5. *Suppose f is a modular form of weight k with real Fourier coefficients. Then $z^{\frac{k+2}{2}} R_k f$ is real valued on $\mathcal{A}$.*

Proof. It is known that if the coefficients $a_n(y)$ of the Fourier expansion of $g(z) = \sum_{n=0}^{\infty} a_n(y) e^{2\pi i n x}$ are real and g satisfies (2), then $z^{\frac{k}{2}} g(z)$ is real valued for $|z| = 1$. See, for example, Proposition 2.1 of [1].

Since f has real Fourier coefficients, then $R_k(f)$ also has real Fourier coefficients. Hence, since $R_k(f)$ transforms like a $k+2$ modular forms, we have that $z^{\frac{k+2}{2}} R_k f$ is real valued on $\mathcal{A}$. □

4. Proof of theorem 1.1

We will prove this theorem by showing that $R_k\left(E_k(e^{i\theta})\right) e^{i\theta \frac{k+2}{2}}$ has $m(k+2)$ zeros in $\mathcal{A}$. Some of the calculations in this section were performed using Mathematica 10.

If we restrict $R_k(E_k(z)) z^{\frac{k+2}{2}}$ to $\mathcal{F}$ and $|z| = 1$, then we have $R_k\left(E_k(z)\right) z^{\frac{k+2}{2}} =$

$$-kie^{i\theta/2} \sum\sum_{\substack{\gcd(c,d)=1\\ c,d\in\mathbb{Z}}} c(ce^{i\theta/2} + de^{-i\theta/2})^{-k-1} + \frac{ke^{i\theta}}{\sin(\theta)} \frac{1}{2} \sum\sum_{\substack{\gcd(c,d)=1\\ c,d\in\mathbb{Z}}} (ce^{i\theta/2} + de^{-i\theta/2})^{-k},$$

where $\frac{\pi}{2} \leq \theta \leq \frac{2\pi}{3}$. Taking the terms with $c^2 + d^2 = 1$, we obtain

$$2k \csc(\theta) \cos\left(\frac{k\theta}{2} + \theta\right).$$

Taking the real part of the terms with $c^2+d^2=2$, we get (using Mathematica)

$$\frac{1}{2}k\csc(\theta)\Big(2^{1-k}(-1)^{-\frac{k}{2}}\sin\left(\frac{\theta}{2}\right)^{-k}-\Big(2\csc\left(\frac{\theta}{2}\right)\Big)^{-k}-\Big(2\csc\left(\frac{\theta}{2}\right)\Big)^{-k}\Big). \tag{10}$$

Adding these two and simplifying, we obtain

$$k\csc(\theta)\Big(2\cos\left(\frac{k\theta}{2}+\theta\right)+(-1)^{\frac{k}{2}}2^{-k}\sin\left(\frac{\theta}{2}\right)^{-k}-\Big(2\csc\left(\frac{\theta}{2}\right)\Big)^{-k}\Big). \tag{11}$$

Lemma 4.1. $R_kE_k(e^{i\theta})e^{i\theta\frac{k+2}{2}} = k\Big(\csc(\theta)\Big(\mathcal{M}_1(\theta)+\mathcal{M}_2(\theta)\Big)+\mathcal{E}_1(\theta)+\mathcal{E}_2(\theta)\Big)$, *where*

$$\begin{aligned}
\mathcal{M}_1(\theta) &= 2\cos\left(\frac{k\theta}{2}+\theta\right),\\
\mathcal{M}_2(\theta) &= (-1)^{\frac{k}{2}}2^{-k}\sin\left(\frac{\theta}{2}\right)^{-k}-\Big(2\csc\left(\frac{\theta}{2}\right)\Big)^{-k},\\
\mathcal{E}_1(\theta) &= -ie^{i\theta/2}\sum_{\substack{\gcd(c,d)=1\\ c^2+d^2\geq 5}}\sum c(ce^{i\theta/2}+de^{-i\theta/2})^{-k-1}\\
\mathcal{E}_2(\theta) &= \frac{e^{i\theta}}{\sin(\theta)}\frac{1}{2}\sum_{\substack{\gcd(c,d)=1\\ c^2+d^2\geq 5}}\sum (ce^{i\theta/2}+de^{-i\theta/2})^{-k}
\end{aligned}$$

Lemma 4.2. *The term $\mathcal{M}_2$ is never positive in the interval $\frac{\pi}{2}\leq\theta\leq\frac{2\pi}{3}$. Furthermore, $|\mathcal{M}_2|\leq 1$*

Proof. Taking the derivative of $\mathcal{M}_2$, we get

$$\frac{k}{2}\left(-\left(-\frac{1}{4}\right)^{\frac{k}{2}}\cos\left(\frac{\theta}{2}\right)\sin^{-k-1}\left(\frac{\theta}{2}\right)-\tan\left(\frac{\theta}{2}\right)\left(2\csc\left(\frac{\theta}{2}\right)\right)^{-k}\right),$$

which is clearly negative on $\frac{\pi}{2}\leq\theta\leq\frac{2\pi}{3}$. Therefore, $\mathcal{M}_2$ is decreasing. We evaluate this at $\theta=\frac{\pi}{2}$ and at $\theta=\frac{2\pi}{3}$.

At $\theta=\pi/2$, $\mathcal{M}_2(\theta)=2^{-\frac{k}{2}}\left((-1)^{\frac{k}{2}}-1\right)$ which is less than or equal to 0 and greater than or equal to $-\frac{1}{4}$.

At $\theta=2\pi/3$, $\mathcal{M}_2(\theta)=\left(-\frac{1}{3}\right)^{\frac{k}{2}}-1$ which is less than or equal to $-\frac{4}{5}$ and greater than or equal to -1.

Hence, $|\mathcal{M}_2|\leq 1$. □

Lemma 4.3. *The number of zeros of $\mathcal{M}_1$ in $\left(\frac{\pi}{2},\frac{2\pi}{3}\right)$ is $m(k+2)$.*

Proof. This argument is similar to the one used by Rankin and Swinnerton-Dyer in [5].

Write $\theta=\frac{2m\pi}{k+2}$ and $k+2=12n+s$, where $s=4,6,8,10,0$ or 14. Then $\mathcal{M}_1(\theta)=2\cos(m\pi)$ which is either 2 or -2 depending on whether m is odd or even. Hence the number of zeros of $\cos\left(\frac{k\theta}{2}+\theta\right)$ in $\left(\frac{\pi}{2},\frac{2\pi}{3}\right)$ is equal to the number of integers in the closed interval $\left[\frac{k+2}{4}, \frac{k+2}{3}\right]$ minus one, which can be verified to be $m(k+2)$. □

Lemma 4.4. *$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2$ has $m(k+2)$ zeros in $\left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$.*

Proof. Since $2\cos(m\pi)$ is either 2 or -2, and $\mathcal{M}_2$ is less than or equal to 1 in absolute value, then $\mathcal{M}_1 + \mathcal{M}_2$ has the same number of zeros as $\mathcal{M}_1$. □

Therefore, it only remains to show that $\mathcal{E}_1(\theta) + \mathcal{E}_2(\theta)$ is less than one. We will use the estimates provided in [5].

Since we know that $R_k\Big(E_k(z)\Big)z^{\frac{k+2}{2}}$ is real valued on the bottom arc of $\mathcal{F}$ by Lemma 3.5, then we have that $R_k\Big(E_k(z)\Big)z^{\frac{k+2}{2}} = \mathrm{Re}\Big[R_k\Big(E_k(z)\Big)z^{\frac{k+2}{2}}\Big]$, which is equal to

$$k\Big(\mathcal{M} + \mathrm{Re}(\mathcal{E}_1) + \mathrm{Re}(\mathcal{E}_2)\Big) \tag{12}$$

Since $\mathrm{Re}[z] \leq |z|$ we will provide bounds for the absolute values of the remaining parts. We will show that the sum of these two absolute values is less than 1.

Lemma 4.5. $\Big|Re(\mathcal{E}_1)\Big| < 0.44$

Proof.

$$\begin{aligned}\mathrm{Re}\Big[-ie^{i\theta/2}\sum_{c^2+d^2\geq 5}\sum c(ce^{i\theta/2}+de^{-i\theta/2})^{-k-1}\Big] &\leq \Big|-ie^{i\theta/2}\sum_{c^2+d^2\geq 5}\sum c(ce^{i\theta/2}+de^{-i\theta/2})^{-k-1}\Big|\\ &= \Big|\sum_{c^2+d^2\geq 5}\sum c(ce^{i\theta/2}+de^{-i\theta/2})^{-k-1}\Big|\\ &\leq 2\Big(4\Big(\frac{5}{2}\Big)^{-k/2} + \sum_{N=10}^{\infty} 5N^{1/2}\Big(\frac{1}{2}N\Big)^{-k/2}\Big)\end{aligned}$$

which is decreasing on k and at $k=8$ is less than 0.44. □

Lemma 4.6. $\Big|Re(\mathcal{E}_2)\Big| < 0.26$

Proof.

$$\begin{aligned}\mathrm{Re}\Big[\frac{e^{i\theta}}{y}\frac{1}{2}\sum_{c^2+d^2\geq 5}\sum(ce^{i\theta/2}+de^{-i\theta/2})^{-k}.\Big] &\leq \Big|\frac{e^{i\theta}}{y}\frac{1}{2}\sum_{c^2+d^2\geq 5}\sum(ce^{i\theta/2}+de^{-i\theta/2})^{-k}.\Big|\\ &= \Big|\frac{e^{i\theta}}{y}\Big|\Big|\frac{1}{2}\sum_{c^2+d^2\geq 5}\sum(ce^{i\theta/2}+de^{-i\theta/2})^{-k}\Big|\\ &\leq \frac{2}{\sqrt{3}}\Big(4\Big(\frac{5}{2}\Big)^{-k/2} + \sum_{N=10}^{\infty} 5N^{1/2}\Big(\frac{1}{2}N\Big)^{-k/2}\Big)\end{aligned}$$

which is decreasing on k and at $k=8$ is less than 0.26. □

Lemma 4.7. $\Big|Re(\mathcal{E}_1) + Re(\mathcal{E}_2)\Big| < 1$

Proof. By Lemma 4.5 and Lemma 4.6, we have that $\Big|\mathrm{Re}(\mathcal{E}_1)+\mathrm{Re}(\mathcal{E}_2)\Big| < .7 < 1$ □

5. Jacobian

Let f be a modular form. Recall the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{13}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{14}$$

Write $f = u + iv$. Then,

$$R_k(f(z)) = 2i\frac{\partial f}{\partial z} + \frac{k}{y}f = -2\frac{\partial}{\partial y}f + \frac{k}{y}f = -2u_y + \frac{k}{y}u + i\Big(-2v_y + \frac{k}{y}v\Big).$$

Let $a = -2u_y + \frac{k}{y}u$ and $b = -2v_y + \frac{k}{y}v$. Then $R_k(f) = a + ib$.

The aim of this section is to calculate the Jacobian of $R_k(f)$.

$J(R_k(f)) = a_x b_y - a_y b_x.$

From the Cauchy-Riemann equations, one can obtain the following useful identities:

$$u_{yy} = -u_{xx}, \quad v_{yy} = -v_{xx}, \tag{15}$$

$$u_{yx} = -v_{xx}, \quad v_{yx} = c_{xx}. \tag{16}$$

Anticipating a simplification, define

$$A = v_{xx}, \quad B = u_{xx}, \quad C = v_y, \quad D = u_y, \tag{17}$$

$$X = 2A + \frac{k}{y}C, \quad Y = 2B + \frac{k}{y}D. \tag{18}$$

Theorem 5.1. *Let $f \in M_k$. Then $Jac(R_k f) = X^2 + Y^2 - \frac{k}{y^2}\Big(uY + vX\Big)$.*

We will separate the calculation in several parts:

5.1. $a_x b_y$.

We begin by calculating a_x:

$$\begin{aligned} a_x &= \frac{\partial}{\partial x}\Big(-2u_y + \frac{k}{y}c\Big) \\ &= -2u_{yx} + \frac{k}{y}u_x \end{aligned}$$

Now b_y.

$$\begin{aligned} b_y &= \frac{\partial}{\partial y}\Big(-2v_y + \frac{k}{y}v\Big) \\ &= -2v_{yy} + \frac{k}{y^2}v + \frac{k}{y}v_y \end{aligned}$$

Multiplying these two:

$$\begin{aligned} a_x b_y &= 4u_{yx}v_{yy} + \frac{2k}{y^2}u_{yx}v - \frac{2k}{y}u_{yx}v_y - \frac{2k}{y}v_{yy}u_x - \frac{k^2}{y^3}u_x v + \frac{k^2}{y^2}u_x v_y \\ &= 4v_{xx}v_{xx} - \frac{2k}{y^2}v_{xx}v + \frac{4k}{y}v_{xx}v_y - \frac{k^2}{y^3}v_y v + \frac{k^2}{y^2}v_y v_y \end{aligned}$$

5.2. $a_y b_x$.

We begin by calculating a_y:

$$\begin{aligned} a_y &= \frac{\partial}{\partial y}\Big(- 2u_y + \frac{k}{y}c\Big) \\ &= -2u_{yy} + \frac{k}{y^2}u + \frac{k}{y}u_y \end{aligned}$$

Now b_x.

$$\begin{aligned} b_x &= \frac{\partial}{\partial x}\Big(- 2v_y + \frac{k}{y}v\Big) \\ &= -2v_{yx} + \frac{k}{y}v_x \end{aligned}$$

Multiplying these two:

$$\begin{aligned} a_y b_x &= 4v_{yx}u_{yy} + \frac{2k}{y^2}v_{yx}u - \frac{2k}{y}v_{yx}u_y - \frac{2k}{y}u_{yy}v_x - \frac{k^2}{y^3}v_x u + \frac{k^2}{y^2}v_x u_y \\ &= -4u_{xx}u_{xx} + \frac{2k}{y^2}u_{xx}u - \frac{4k}{y}u_{xx}u_y + \frac{k^2}{y^3}u_y u - \frac{k^2}{y^2}u_y u_y \end{aligned}$$

5.3. $a_x b_y - a_y b_x$.

Adding these, we obtain

$$4(v_{xx}v_{xx} + u_{xx}u_{xx}) - \frac{2k}{y^2}\Big(v_{xx}v + u_{xx}u\Big) + \frac{4k}{y}\Big(v_{xx}v_y + u_{xx}u_y\Big) - \frac{k^2}{y^3}\Big(v_y v + u_y u\Big) + \frac{k^2}{y^2}\Big(v_y v_y + u_y u_y\Big)$$

Then $\mathrm{Jac}(R_k f) = X^2 + Y^2 - \frac{k}{y^2}\Big(uY + vX\Big)$. Writing a_y and b_y in terms of X, Y, u and v, we get

$$b_y = X - \frac{k}{y^2}v, \tag{19}$$

$$a_y = Y - \frac{k}{y^2}u. \tag{20}$$

Then, when $a = b = 0$ we have

$$\mathrm{Jac}(R_k f) = \Big(b_y + \frac{v_y}{y}\Big)^2 - \frac{v_y^2}{y^2} + \Big(a_y + \frac{u_y}{y}\Big)^2 - \frac{u_y^2}{y^2}. \tag{21}$$

If f is the weight k Eisenstein series, then,

$$v = -\frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)e^{-2\pi ny}\sin(2\pi nx) \tag{22}$$

$$u = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)e^{-2\pi ny}\cos(2\pi nx). \tag{23}$$

Question 5.1. If $f \in M_k$, does the Jacobian of $R_k f$ vanish?

6. Conjectures and Future Work

Let $R_k^j = R_{k+2j-2} \circ \cdots \circ R_{k+2} \circ R_k$ and suppose f is a modular form of weight k. Then,

$$R_k^j(f) = \sum_{r=0}^{j}(-1)^r 4^r \binom{j}{r}\frac{(k+r)_{j-r}}{y^{j-r}}D^r f \tag{24}$$

where $(a)_m = a(a+1)\cdots(a+m-1)$ is the Pochhammer symbol. See (4.15) of [4] for details.

Also, it can be shown that

$$E_k^{(j)} = \frac{(-1)^j (k)_j}{2(2\pi i)^j}\sum\sum_{\substack{\gcd(c,d)=1\\ c,d\in\mathbb{Z}}} c^j (cz+d)^{-(k+j)} \tag{25}$$

Hence, in the case of Eisenstein series, we have

$$R_k^j(E_k(z)) = \frac{(k)_j}{2y^j}\sum_{r=0}^{j}(-2iy)^r\binom{j}{r}\sum\sum c^r(cz+d)^{-(k+r)} \tag{26}$$

Conjecture 6.1. *All of the zeros of $R_k(E_k(z))$ inside the fundamental domain lie on $\mathcal{A}$.*

Conjecture 6.2. *$R_k^j(E_k)$ has the same amount of zeros as E_{k+2j}. Furthermore, all of the zeros of $R_k^j(E_k)$ inside the fundamental domain lie on $\mathcal{A}$.*

Conjecture 6.3. *$ReE_k(z) > 0$ on the interior of $\mathcal{F}$.*

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