Refining Fewnomial Theory for 2×2 Systems

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History

Descartes' Rule of Signs (17^{th} century)

If $f(x) := c_1 x^{a_1} + \cdots + c_T x^{a_T} \in \mathbb{R}[x, x^{-1}]$ and $(a_1 < \cdots < a_t)$, then the number of positive roots (counting multiplicity) is less than or equal to the number of sign alternations in $(c_1, \cdots, c_T)$.

- Direct consequence is that the maximum finite number of positive roots is $(T - 1)$
- Relating Descartes' Rule to multivariable systems of polynomials remains a difficult open problem

Definitions

Definition

We define a **2×2 System** as a system of two polynomials and two variables.

Definition

We define a systems of two variables where one is a trinomial and the other is an *m*-nomial as a **System of Type (3,m)**.

Example:

$$\beta + x^{r_2} y^{s_2} + x^{r_3} y^{s_3}$$

$$\alpha_1 + \alpha_2 x^{a_2} y^{b_2} + \cdots + \alpha_m x^{a_m} y^{b_m}$$

where $\beta, \alpha_1, \ldots, \alpha_m \in \mathbb{R}$.

Goals

We look at systems of type (3,m):

$$\beta + x^{r_2}y^{s_2} + x^{r_3}y^{s_3}$$
$$\alpha_1 + \alpha_2 x^{a_2}y^{b_2} + \cdots + \alpha_m x^{a_m}y^{b_m}$$

where $\beta, \alpha_1, \ldots, \alpha_m \in \mathbb{R}$.

- The maximum finite number of roots in $\mathbb{R}^2_+$ of systems of type $(3, m)$ is known to lie between $2m - 1$ and $\frac{2}{3}m^3 + 5m$

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where $\beta, \alpha_1, \ldots, \alpha_m \in \mathbb{R}$.

- The maximum finite number of roots in $\mathbb{R}^2_+$ of systems of type $(3, m)$ is known to lie between $2m - 1$ and $\frac{2}{3}m^3 + 5m$
- We want to tighten current bounds
- We want to construct new extremal examples of minimal height (simpler examples)

Techniques

Rolle's Theorem

If $f : [a, b] \longrightarrow \mathbb{R}$ is continuous and differentiable, and $f(a) = f(b)$, then there is a $c \in (a, b)$ such that $f'(c) = 0$.

- Techniques applied to this problem have been variants of Rolle's Theorem and a result of Polya on the Wronskian
- We will consider intersections of convex arcs

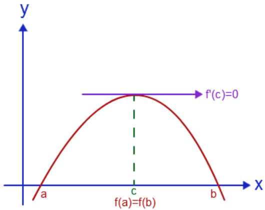

Figure: Rolle's Theorem

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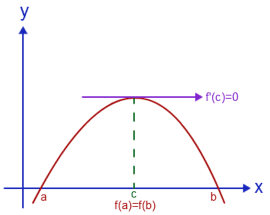

Figure: Rolle's Theorem

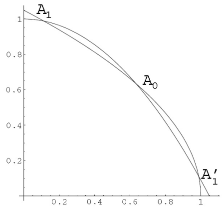

Figure: Haas, 2000

Big Picture

- New systems help by giving insight into fewnomial theory
- New systems also lead to derivations of new facts

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- New systems also lead to derivations of new facts

- The idea so to create a system of type $(3, m)$ with $2m - 1$ roots in $\mathbb{R}^2_+$ in order to get a system of type $(3, m + 1)$ with $2(m + 1) - 1$ roots in $\mathbb{R}^2_+$
- So we will start with a system of type $(3, 3)$ to construct a system of type $(3, 4)$

Big Picture

In 2000, Haas found the first 2×2 system of type $(3,3)$ with 5 roots in $\mathbb{R}^2_+$

$$y^{106} + 1.1x^{53} - 1.1x$$
$$x^{106} + 1.1y^{53} - 1.1y$$

In 2007, the simplest 2×2 system of type (3,3) with 5 roots in $\mathbb{R}^2_+$, discovered by Dickenstein, Rojas, Rosek, and Shih was found:

$$x^6 + \frac{44}{31}y^3 - y$$
$$y^6 + \frac{44}{31}x^3 - x$$

Small Picture

We specifically look at the following system:

$$f(x_1, x_2) := x_1^5 - \frac{49}{95}x_1^3 x_2 + x_2^6$$
$$g(x_1, x_2) := x_2^5 - \frac{49}{95}x_1 x_2^3 + x_1^6$$

- We verify we have 5 roots in $\mathbb{R}_+^2$
- We reduce the system
- Construct a 2×2 system of type $(3, 4)$ by adding a monomial term

Finding Roots

We start with

$$f(x_1, x_2) := x_1^5 - \frac{49}{95}x_1^3x_2 + x_2^6$$
$$g(x_1, x_2) := x_2^5 - \frac{49}{95}x_1x_2^3 + x_1^6$$

By rescaling and performing a change of variables, we got

$$r(u, v) := u - \frac{49}{95} + v$$
$$s(u, v) := u^{\frac{1}{7}}v^{\frac{3}{7}} - \frac{49}{95} + u^{\frac{16}{7}}v^{\frac{-1}{7}}$$

where $u = x_1^2x_2^{-1}$ and $v = x_1^{-3}x_2^5$

Finding Roots

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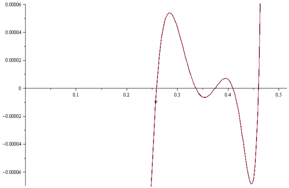
Setting $r = s = 0$, we get the following algebraic function:

$$G(u) := u^{\frac{1}{7}} \left(\frac{49}{95} - u\right)^{\frac{3}{7}} - \frac{49}{95} + u^{\frac{16}{7}} \left(\frac{49}{95} - u\right)^{\frac{-1}{7}} = 0$$



Finding Roots

We care about roots that lie in the interval $(0, \frac{49}{95})$. Why?

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Recall

We obtained $G(u)$ by setting $r = s = 0$. So

$$r(u, v) := u - \frac{49}{95} + v = 0 \Rightarrow v = \frac{49}{95} - u$$

- So the roots of G(u) that lie in $(0, \frac{49}{95})$ imply that v is also positive.

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- This implies $x_1, x_2 > 0$

Finding Roots

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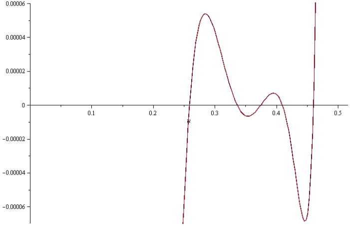
Why do we care about the roots at all?

- Finding roots will give us Regions of Interest to insert a "hump" that yields 7 intersections with $G(u)$
- How do we find these roots?



Finding Roots

Definition

Given any $d, e \in \mathbb{N}$ and $f, g \in \mathbb{C}[x]$ with $\deg(f) \leq d$ and $\deg(g) \leq e$, the **Sylvester Matrix** of (f, g) of format (d, e) is:

$$\mathrm{SYL}_{(d,e)}(f,g) = \begin{pmatrix} a_0 & a_1 & \cdots & a_d & 0 & \cdots & 0 \\ 0 & a_0 & a_1 & \cdots & a_d & \cdots & 0 \\ \vdots & \ddots & \ddots & & & \ddots & \\ 0 & \cdots & 0 & a_0 & a_1 & \cdots & a_d \\ b_0 & b_1 & \cdots & b_e & 0 & \cdots & 0 \\ 0 & b_0 & b_1 & \cdots & b_e & \cdots & 0 \\ \vdots & \ddots & \ddots & & & \ddots & \\ 0 & \cdots & 0 & b_0 & b_1 & \cdots & b_e \end{pmatrix}$$

Figure: Sylvester Matrix of (f, g) of format (d, e)

Finding Roots

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Definition

The **Resultant** of f and g (denoted $\text{Res}_{(d,e)}(f, g)$) is the determinant of their Sylvester Matrix.

Finding Roots

- We have

$$r(u,v) := u - \frac{49}{95} + v$$
$$s(u,v) := u^{\frac{1}{7}} v^{\frac{3}{7}} - \frac{49}{95} + u^{\frac{16}{7}} v^{\frac{-1}{7}}$$

- Set $u = p^7$ and $v = q^7$ and multiply $z(p,q)$ by q

$$t(p,q) := p^7 - \frac{49}{95} + q^7$$
$$z(p,q) := pq^4 - \frac{49}{95} q + p^{16}$$

- Now we get the resultant of t and z with respect to q to find the roots

Finding Roots

- The resultant yields the following polynomial

$$p^{112} - \frac{823543}{857375}p^{56} + \cdots - \frac{33232930569601}{6634204312890625}$$

Finding Roots

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$$p^{112} - \frac{823543}{857375}p^{56} + \cdots - \frac{33232930569601}{663420431289062 5}$$

- Recall we had $u = p^7$. So we substitute $p = u^{\frac{1}{7}}$.

$$u^{16} - \frac{823543}{857375}u^{8} + \cdots - \frac{33232930569601}{6634204312890625}$$

- This is an easier polynomial to compute roots

Humps and Bumps

$$G(u) := u^{\frac{1}{7}}\left(\frac{49}{95}-u\right)^{\frac{3}{7}} - \frac{49}{95} + u^{\frac{16}{7}}\left(\frac{49}{95}-u\right)^{\frac{-1}{7}} = 0$$

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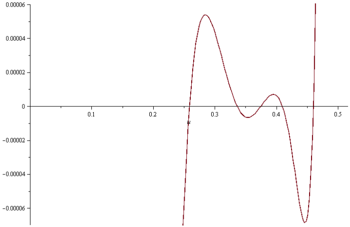

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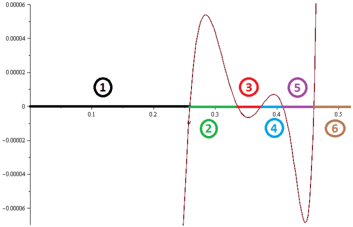

Constructing New Systems

$$G(u) := u^{\frac{1}{7}}\left(\frac{49}{95} - u\right)^{\frac{3}{7}} - \frac{49}{95} + u^{\frac{16}{7}}\left(\frac{49}{95} - u\right)^{\frac{-1}{7}} = 0$$

- So where do the humps come from?
- We use the monomial term

$$H(u) := cu^{a}\left(\frac{49}{95} - u\right)^{b}$$

- For what (a, b, c) does $G(u)$ and $H(u)$ have 7 intersections

Humps and Bumps

$$H(u) := cu^a \left(\frac{49}{95} - u\right)^b$$

How do we choose (a, b, c)?

- We want to insert a hump in some interval (i_1, i_2)
- We want the peak to be at the midpoint $\left(\frac{i_1+i_2}{2}\right)$
- We want the the inflection points to be at the endpoints i_1, i_2

Humps and Bumps

$$H(u) := cu^a \left(\frac{49}{95} - u\right)^b$$

How do we choose (a, b, c)?

- By taking some derivatives and with some algebra we find that

$$a = \frac{m^2}{d^2}\left(1 - \frac{95m}{49}\right) + \frac{95m}{49}$$

where $m = \frac{i_1+i_2}{2}$ and $d = \frac{i_2-i_1}{2}$

Humps and Bumps

$$H(u) := cu^a \left(\frac{49}{95} - u\right)^b$$

How do we choose (a, b, c)?

- We also get

$$b = \frac{49a}{95m} - a$$

where $m = \frac{i_1+i_2}{2}$ and

$$c = h \cdot \left(\frac{a+b}{49/95}\right)^{a+b} \cdot \frac{1}{a^a b^b}$$

where h is the desired height of the peak of $H(u)$

Once we get a $H(u)$ that intersects $G(u)$, what is next?

- We let $G_2(u) = G(u) - H(u)$

$$G_2(u) = u^{\frac{1}{7}}\left(\frac{49}{95} - u\right)^{\frac{3}{7}} - \frac{49}{95} + u^{\frac{16}{7}}\left(\frac{49}{95} - u\right)^{\frac{-1}{7}} - cu^a\left(\frac{49}{95} - u\right)^b$$

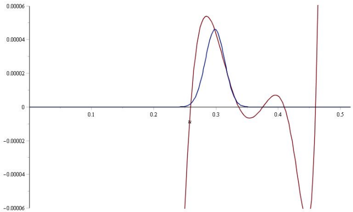

Figure: $G(u)$ is red; $H(u)$ is blue

Constructing New Examples

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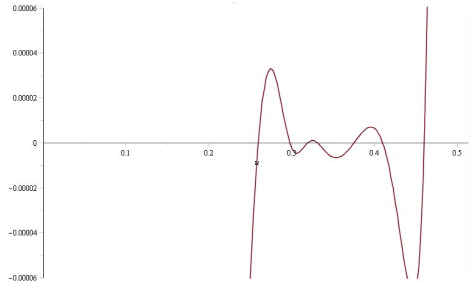

Figure: $G_2(u)$

Constructing New Examples

How do we create a new system?

$$G_2(u) = u^{\frac{1}{7}} \left(\frac{49}{95} - u\right)^{\frac{3}{7}} - \frac{49}{95} + u^{\frac{16}{7}} \left(\frac{49}{95} - u\right)^{\frac{-1}{7}} - cu^a \left(\frac{49}{95} - u\right)^b$$

- We undo the substitution to get a new system

$$r_2(u, v) := u - \frac{49}{95} + v$$

$$s_2(u, v) := u^{\frac{1}{7}} v^{\frac{3}{7}} - \frac{49}{95} + u^{\frac{16}{7}} v^{\frac{-1}{7}} - cu^a v^b$$

- Undo change of variables to get 2×2 system of type $(3, 4)$ with 7 roots in $\mathbb{R}_+^2$

Finding More examples

- We started off by finding two humps for regions 2-5
  - One centered between two endpoints of the region
  - One centered on actual peak of that region
- We then found examples in Regions 1 & 6
- We found examples with humps closer to endpoints

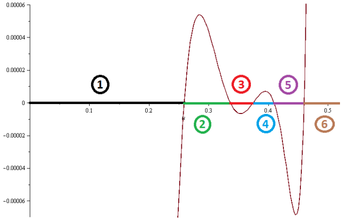

Figure: Regions of Interest

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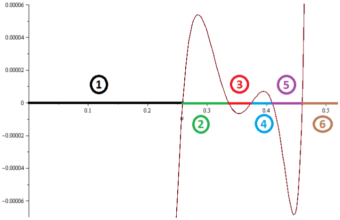

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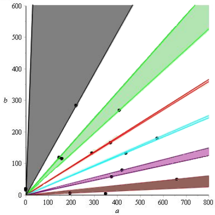
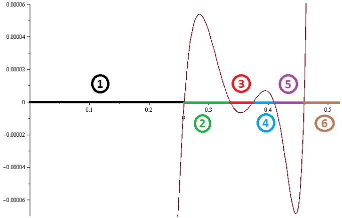


Figure: Results

Finding More examples

Please note...

- Scalar multiples work too!
- Colored areas yield more possible examples



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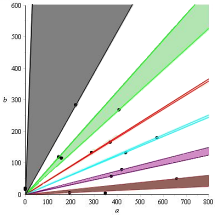

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Quest to a Simple Example

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In 2007, the first known 2×2 system of type $(3, 4)$ with 7 roots in $\mathbb{R}^2_+$ was discovered by Gomez, Niles, and Rojas

$$x^6 + \frac{44}{31}y^3 - y$$

$$y^{14} + \frac{44}{31}x^3y^8 - xy^8 + 1936254x^{133}$$

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GOAL! We found a new example!

Quest to a Simple Example

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My example

$$x^5 - \frac{49}{95}x^3y + y^6$$

$$x^{33}y^5 - \frac{49}{95}y^3x^{34} + x^{39} + 5807y^{62}$$

Quest to a Simple Example

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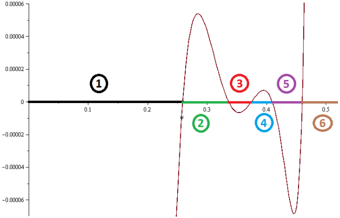

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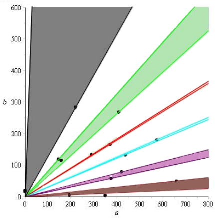

Figure: Results

New Direction

There are more systems to look at!

$$1 + x^4 - \frac{10}{17}x^5y^2$$
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Preliminary Result:

$$1 + x^4 - \frac{10}{17}x^5y^2$$
$$1 + y^4 - \frac{10}{17}x^2y^5 + 102000x^{-94}y^{-35}$$

New Direction

We now look to prove the following:

Conjecture

Let us fix an integer k. Then the maximum number of roots of

$$\sum_{i=1}^{k} c_i u^{a_i}(1-u)^{b_i} \text{ in } (0,1)$$

(over all real a_i, b_i, and c_i) is $O(k)$.