On the Zeroes of Half Integral Weight Eisenstein Series of $\Gamma_0(4)$

Samantha Moore

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Background

Definition

$$\mathbf{\Gamma_0(4)} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1, c \equiv 0 \bmod 4 \right\}$$

Definition

The Eisenstein series of weight $\frac{k}{2}$ for each of the cusps of $\Gamma_0(4)$ are modular forms defined as:

- $\mathbf{E}_\infty(\mathbf{z}) = e^{\frac{\pi i k}{4}} \sum_{(2c,d)=1, c>0} \frac{G(\frac{-d}{4c})^k}{(4cz+d)^{k/2}}$.
- $\mathbf{E_0}(\mathbf{z}) = \sum_{(u,2v)=1, u>0} \frac{(\frac{-v}{u})\epsilon_u^k}{(uz+v)^{k/2}}$.
- $\mathbf{E}_{\frac{1}{2}}(\mathbf{z}) = e^{\frac{-\pi i k}{4}} \sum_{(2c,d)=1, d>0} \frac{G(\frac{d-2c}{8d})^k}{(dz+c)^{k/2}}$.

Project Goal

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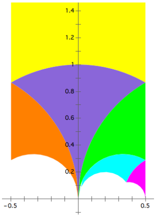
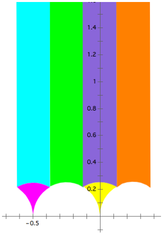
I wish to determine the location of the zeroes of the Eisenstein series E_∞ of $\Gamma_0(4)$.

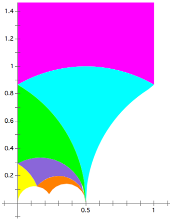
Fundamental Domains



F_∞



F_0



$F_{\frac{1}{2}}$

Zeroes of $\Gamma_0(4)$

Theorem

For k sufficiently large, all but at most $O(\sqrt{k \log k}) + 4$ zeroes of $E_\infty(z, k)$ lie on the lines $x = -\frac{1}{2}$ of F_0 and $x = \frac{1}{2}$ of $F_{\frac{1}{2}}$.

Proof Overview

- Show that $e^{\frac{\pi ik}{4}} E_0(-\frac{1}{2} + iy, k)$ is a real valued function
- Find a real valued trigonometric approximation of $e^{\frac{\pi ik}{4}} E_0(-\frac{1}{2} + iy, k)$, which we denote as $e^{\frac{\pi ik}{4}} M_0$
- Bound the error of this approximation for large k and $y \leq \frac{c\sqrt{k}}{\sqrt{\log k}}$, where $c \leq 1$ is a constant
- Use the Intermediate Value Theorem to determine zeroes of M_0
- By our bounds on the error of M_0 in relation to $E_0(-\frac{1}{2} + iy, k)$, we prove that each of the zeroes of M_0 correspond to a zero of $E_0(-\frac{1}{2} + iy, k)$ and thus a zero of $E_\infty(z, k)$.

Show that $e^{\frac{\pi i k}{4}} E_0(-\frac{1}{2} + iy, k)$ is a real valued function

- We will use the Fourier expansion of $E_0(z, k)$, which is defined as

$$E_0(z, k) = 2^{\frac{k}{2}} \sum_{\ell=1}^{\infty} b_\ell q^\ell$$

where $q = e^{2\pi i z}$ and

$$b_\ell = \frac{\pi^{\frac{k}{2}} \ell^{\frac{k}{2}-1}}{\Gamma(\frac{k}{2}) e^{\frac{\pi i k}{4}}} \sum_{n_0 > 0 \ odd} \epsilon_n^k n^{-\frac{k}{2}} \sum_{j=0}^{n-1} \left(\frac{j}{n}\right) e^{-\frac{2\pi i \ell j}{n}}.$$

Show that $e^{\frac{\pi ik}{4}} E_0(-\frac{1}{2} + iy, k)$ is a real valued function

- Case 1: When ℓ is squarefree, Koblitz simplifies b_ℓ to

$$b_\ell = \frac{\pi^{\frac{k}{2}} \ell^{\frac{k}{2}-1}}{\Gamma(\frac{k}{2}) e^{\frac{\pi ik}{4}}} \sum_{n_0>0 \ odd} \sum_{n_1 | \ell, n_1 odd} \epsilon_{n_0 n_1^2}^{k+1} (n_0 n_1^2)^{-\frac{k}{2}} \left(\frac{-\ell}{n_0} \right) \sqrt{n_0} \mu(n_1) n_1$$

where

$$\mu(n_1) = \begin{cases} 0, & n_1 \text{ not squarefree} \\ (-1)^r, & n_1 \text{ is the product of r distinct primes} \end{cases}.$$

- Note that $\epsilon_{n_0 n_1^2}^{k+1} = \pm 1$ as k is odd.
- Thus, every part of b_ℓ is real except for the factor $e^{-\frac{\pi ik}{4}}$.

Show that $e^{\frac{\pi i k}{4}} E_0(-\frac{1}{2} + iy, k)$ is a real valued function

- Case 2: (ℓ not squarefree. Let $\ell = p^{2v}\ell_0$ and $p^2 \nmid \ell_0$. By Koblitz,

$$\frac{b_\ell}{b_{\ell_0}} = \begin{cases} 2^{(k-2)v}, & \text{p=2} \\ \sum_{h=0}^{v} p^{h(k-2)}, & \text{p odd prime } p \mid \ell_0 \\ \sum_{h=0}^{v} p^{h(k-2)} - \chi_{(-1)^\lambda \ell_0}(p) p^{\lambda-1} \sum_{h=0}^{v} p^{h(k-2)}, & \text{p odd prime } p \nmid \ell_0. \end{cases}$$

  where $\lambda = \frac{k-1}{2}$ and $\chi_{(-1)^\lambda \ell_0} = \left(\frac{-1}{p}\right)^\lambda \left(\frac{\ell_0}{p}\right)$.

- Thus, $b_\ell = A b_{\ell_0}$ where $A \in \mathbb{R}$

Show that $e^{\frac{\pi i k}{4}} E_0(-\frac{1}{2} + iy, k)$ is a real valued function

- Remember that $\ell = p^{2v}\ell_0$. We could continue pulling factors out of ℓ until we arrive at a squarefree value, ℓ_*. This would give us a chain of equivalencies, $b_\ell = Ab_{\ell_0} = ABb_{\ell_1} = ... = AB...Nb_{\ell_*}$ where each scalar is a real constant.
- Thus, $\ell = Cb_{\ell_*}$ where $C \in \mathbb{R}$. Furthermore, $e^{\frac{\pi i k}{4}} b_\ell = Ce^{\frac{\pi i k}{4}} b_{\ell_*}$ By the first case, the right side is now real valued, and thus the left side must also be real

Show that $e^{\frac{\pi i k}{4}} E_0(-\frac{1}{2} + iy, k)$ is a real valued function

- Returning to the Fourier expansion, we now have

$$e^{\frac{\pi i k}{4}} E_0(z, k) = 2^{\frac{k}{2}} \sum_{\ell=1}^{\infty} e^{\frac{\pi i k}{4}} b_\ell q^\ell$$

where $q = e^{2\pi i z}$.

Approximating $e^{\frac{\pi i k}{4}} E_0(-\frac{1}{2} + iy, k)$

- $E_0(z) = \sum_{(u,2v)=1, u>0} \frac{(\frac{-v}{u})\epsilon_u^k}{(uz+v)^{k/2}}$.
- We aim to find a finite approximation for this infinite sum that is accurate for k large enough. Thus, consider the following terms

$$u = 1, v = 0 : \frac{1}{z^{\frac{k}{2}}} = \frac{1}{(-\frac{1}{2} + iy)^{\frac{k}{2}}} = \frac{1}{(re^{i(\pi - \delta)})^{\frac{k}{2}}}$$

$$u = 1, v = 1 : \frac{1}{(z+1)^{\frac{k}{2}}} = \frac{1}{(\frac{1}{2} + iy)^{\frac{k}{2}}} = \frac{1}{(re^{i\delta})^{\frac{k}{2}}}$$

Note that $\delta = arctan(2y)$. Let

$$M_0 = \frac{1}{(re^{i\delta})^{\frac{k}{2}}} + \frac{i^k}{(re^{i(\pi-\delta)})^{\frac{k}{2}}}.$$

Approximating $e^{\frac{\pi i k}{4}} E_0(-\frac{1}{2} + iy, k)$

- We convert to a trigonometric function by the identity $e^{ix} = \cos(x) + i\sin(x)$. From here, by using trigonometric identities and simplifying, we find that

$$M_0 = r^{-\frac{k}{2}} e^{-\frac{\pi i k}{4}} \sqrt{2} \begin{cases} \cos\left(\frac{\delta k}{2} - \frac{\pi}{4}\right), & k \equiv 1 \mod 4 \\ \cos\left(\frac{\delta k}{2} + \frac{\pi}{4}\right), & k \equiv 3 \mod 4. \end{cases}$$

- Note that $e^{\frac{\pi i k}{4}} M_0$ is real valued.

Approximating $e^{\frac{\pi ik}{4}} E_0(-\frac{1}{2} + iy, k)$

- The following two sums include all of the terms left to be bounded:

$$J_1 = \sum_{v \neq 0,1} \frac{\left(\frac{-v}{1}\right)\epsilon_1^k}{(z+v)^{\frac{k}{2}}} \qquad J_2 = \sum_{(u,2v)=1, u>1} \frac{\left(\frac{-v}{u}\right)\epsilon_u^k}{(uz+v)^{\frac{k}{2}}}.$$

- Using tools such as the triangle inequality, bounding sums by integrals, etc. we find that $|J_1| = o(1)$, $|J_2| << \left(\frac{8}{81}\right)^{\frac{k}{4}}$ when k large and $y \leq \frac{c\sqrt{k}}{\sqrt{\log k}}$
- Note that $e^{\frac{\pi ik}{4}}(J_1 + J_2)$ is real valued.

Use the Intermediate Value Theorem to determine zeroes of M_0

- Recall that

$$e^{\frac{\pi ik}{4}} M_0 = r^{-\frac{k}{2}} \begin{cases} \sqrt{2}\cos\left(\frac{\delta k}{2} - \frac{\pi}{4}\right), & k \equiv 1 \mod 4 \\ \sqrt{2}\cos\left(\frac{\delta k}{2} + \frac{\pi}{4}\right), & k \equiv 3 \mod 4 \end{cases}$$

  is a real valued function (where $\delta = \arctan 2y$).

- Note that, as M_0 is a valid approximation for $\frac{1}{2} \leq y \leq \frac{c\sqrt{k}}{\sqrt{\log k}}$, we can bound δ to the interval $\frac{\pi}{4} \leq \delta \leq \arctan \frac{2c\sqrt{k}}{\sqrt{\log k}}$. From here on, we use the notation $y_{max} = \frac{c\sqrt{k}}{\sqrt{\log k}}$.

Use the Intermediate Value Theorem to determine zeroes of M_0

- We wish to find sample points of this function that have the greatest absolute value. Thus, for $k \equiv 1 \mod 4$, we want $\frac{\delta k}{2} - \frac{\pi}{4} = n\pi$ for some $n \in \mathbb{N}$. Solving for δ, we find $\delta = \frac{2\pi n}{k} + \frac{\pi}{2k}$.
- Substituting this into our interval for δ above, we get $\frac{\pi}{4} \leq \frac{2\pi n}{k} + \frac{\pi}{2k} \leq \arctan(2y_{max})$.

Use the Intermediate Value Theorem to determine zeroes of M_0

- Next we solve for n, getting

$$\frac{k}{8} - \frac{1}{4} \leq n \leq \frac{k}{2\pi} \arctan\left(2y_{max}\right) - \frac{1}{4}.$$

- Using some properties of $\arctan(x)$, we can simplify this to

$$\frac{k}{8} - \frac{1}{4} \leq n \leq \frac{k-1}{4} - O(\frac{k}{y_{max}}).$$

Use the Intermediate Value Theorem to determine zeroes of M_0

- As the sign of $\cos\left(\frac{\delta k}{2} - \frac{\pi}{4}\right)$ changes every time n increases, this describes approximately $\frac{k}{8} - O(\frac{k}{y_{max}}) - 1$ sign changes.
- By the Intermediate Value Theorem, there must be a zero of $e^{\frac{\pi ik}{4}} M_0$ between each of these sign changes, so we have found approximately$\frac{k}{8} - O(\frac{k}{y_{max}}) - 2$ zeroes.

Proving the Main Theorem

- Recall that $e^{\frac{\pi ik}{4}}E_0(z,k) = e^{\frac{\pi ik}{4}}M_0 + e^{\frac{\pi ik}{4}}(o(1) + c_1(\frac{8}{81})^{\frac{k}{4}})$ for $x = -\frac{1}{2}$ and k large. From this, each sign change of $e^{\frac{\pi ik}{4}}M_0$ found above also corresponds to a sign change of $e^{\frac{\pi ik}{4}}E_0(z,k)$.
- Therefore, by the IVT, we have found approximately $\frac{k}{8} - O(\frac{k}{y_{max}}) - 2$ zeroes of $e^{\frac{\pi ik}{4}}E_0(z,k)$ when k is large.

Proving the Main Theorem

- Repeating this entire process for $E_{\frac{1}{2}}(z, k)$, we find a total of approximately $\frac{k}{4} - O(\frac{k}{y_{max}}) - 4$ zeroes of $E_\infty(z, k)$.
- By the valence formula for $E_\infty(z, k)$, there are at most $\lfloor \frac{k}{4} \rfloor$ zeroes. Therefore we are missing approximately $O(\frac{k}{y_{max}}) + 4 = O(\sqrt{k \log k}) + 4$ zeroes.

Thank you for listening.