Every Neural Code Can Be Realized by Convex Sets

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Abstract

Place cells are neurons found in some mammals that fire based on the animal's location in their environment. Each place cell fires in an approximately convex region called its receptive field, a subset of a Euclidean space. From the intersections of these receptive fields, a corresponding binary code is extracted. This leads us to ask: is every binary code realizable by convex sets in a Euclidean space? We answer this question in the affirmative via a construction for a convex realization of an arbitrary code $\mathcal{C}$ in $\mathbb{R}^{d-1}$, where d is the number of nonempty codewords in $\mathcal{C}$. We then explore the relationship between a code and its minimal embedding, the smallest dimension in which it is convex realizable. We provide a sufficient condition for the minimal embedding dimension of a convex open code in dimension 2 and conclude by proving that, in some cases, the dimension of the construction is the minimal embedding dimension of a code.

1 Introduction

Place cells were first discovered in 1971 by John O'Keefe, an accomplishment for which he shared the 2014 Nobel Prize in Physiology or Medicine. Place cells are neurons that fire when an animal is in a particular place relative to their environment and thus allow the animal to identify where it is spatially. These neurons fire in approximately convex regions called receptive fields. From the intersections of the receptive fields, we obtain a binary code called the neural code [3].

Definition 1. A *neural code* on n neurons is a set of binary strings $\mathcal{C} \subseteq \{0,1\}^n$. The elements of $\mathcal{C}$ are called *codewords.*

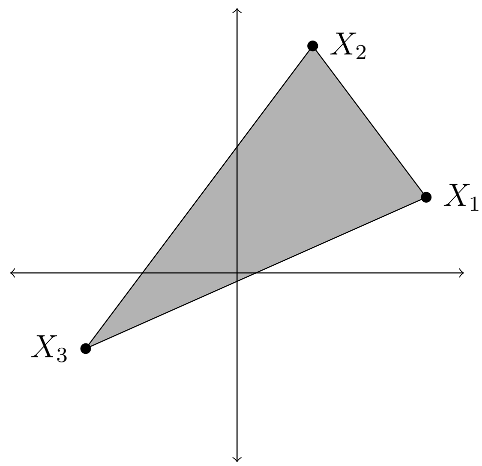
For simplicity, we will refer to a codeword by its support set. For example, the codeword 011 will be referred to as 23.

Definition 2. A code $\mathcal{C} \subseteq \{0,1\}^n$ is *convex* if there exists a set of convex sets, not necessarily open or closed, $\mathcal{U} = \{U_1, ..., U_n\}$ in $\mathbb{R}^d$ such that $\mathcal{C} = \mathcal{C}(\mathcal{U}) := \{\sigma \in [n] \mid U_\sigma \setminus \bigcup_{j \in [n] \setminus \sigma} U_j \neq \emptyset\}$. If such $\mathcal{U}$ exists, then we say that $\mathcal{C}$ is *convex realizable.* The minimal d such that $\mathcal{C}$ is convex realizable in $\mathbb{R}^d$ is called the *minimal embedding dimension.*

Definition 3. If a code $\mathcal{C}$ is convex realizable by a set $\mathcal{U}$ and each $U_i \in \mathcal{U}$ is open convex, we say that $\mathcal{C}$ is *open convex*. Similarly, if a code $\mathcal{C}$ is convex realizable by a set $\mathcal{U}$ and each $U_i \in \mathcal{U}$ is closed convex, we say that $\mathcal{C}$ is *closed convex*.

Definition 4. Let $X_1, X_2, \dots, X_n$ be subsets of $\mathbb{R}^d$. The *convex hull* of $\{X_1, X_2, \dots, X_n\}$ is the smallest convex set in $\mathbb{R}^d$ containing $\{X_1, X_2, \dots, X_n\}$, denoted by $\text{conv}\{X_1, X_2, \dots, X_n\}$.

Example 1. Let $X_1 = (2.5, 1), X_2 = (1, 3)$, and $X_3 = (-2, -1)$ be points in $\mathbb{R}^2$. Then, $\text{conv}(X_1, X_2, X_3) = \text{conv}\{(2.5, 1), (1, 3), (-2, -1)\}$ is depicted below:



One of the primary goals of this area of research is to determine which codes are convex realizable. Much work has been done on determining which codes are open convex and closed convex [3] [4]. Cruz et al. showed that all max-intersection complete codes are both open and closed convex [2]. Another area of significant interest has been the minimal embedding dimension of convex codes, that is, the smallest dimension for which there exists a realization of the code. Mulas and Tran completely characterized the minimal embedding dimensions of open connected codes [5]. However, less work has been done investigating convex codes without regard to openness nor closedness.

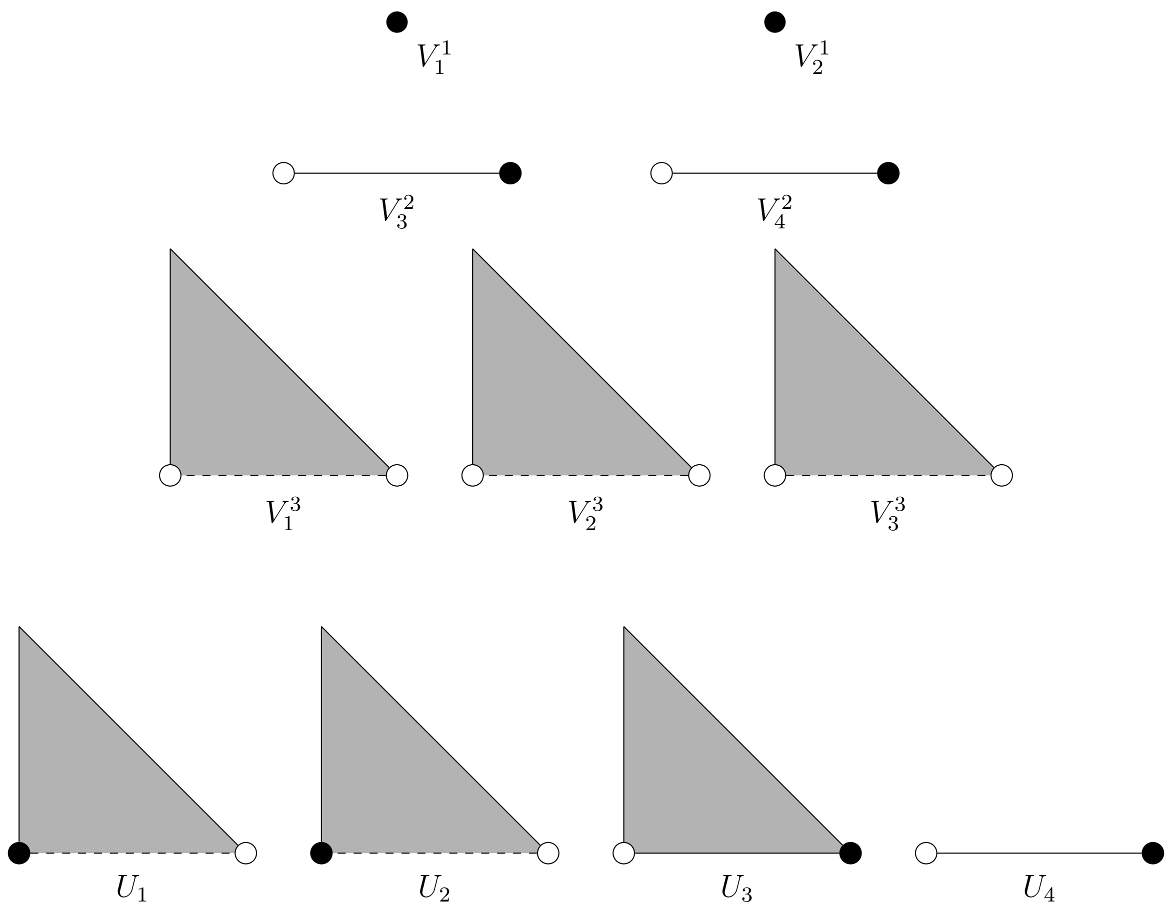
If a code is open convex or closed convex, then by definition it is convex so the set of codes which are only convex contains all codes which are open convex and closed convex. Some have speculated that all locally great codes are convex [1] while others have speculated that in fact all codes are convex. We will show that every neural code is convex. That is, every neural code is realizable by a set $\mathcal{U}$ where each $U_i \in \mathcal{U}$ is convex but not necessarily open or closed.

2 Main Results

Our primary result is a construction of a convex realization of an arbitrary code $\mathcal{C}$ in $\mathbb{R}^{d-1}$ where d is the number of nonempty codewords in $\mathcal{C}$. We will begin with two examples of the construction, followed by a proof of the construction in Theorem 1. Following this result, the remainder of the paper explores the relationship between a code and its minimal embedding dimension. In Theorem 2, we give a sufficient condition for the the minimal embedding dimension of a convex open code to be 2. Finally, we conclude by proving in Theorem 3 that, for a certain class of codes, the dimension of the construction in Theorem 1 is the minimal embedding dimension of the code.

Example 2. Consider the code $\mathcal{C} = \{\emptyset, 12, 34, 123\}$. Figure 1 displays the convex sets U_1, U_2, U_3, and U_4 which realize $\mathcal{C}$ as well as the V^i_j that are used in the construction of the U_i's (see proof of in Theorem 1).

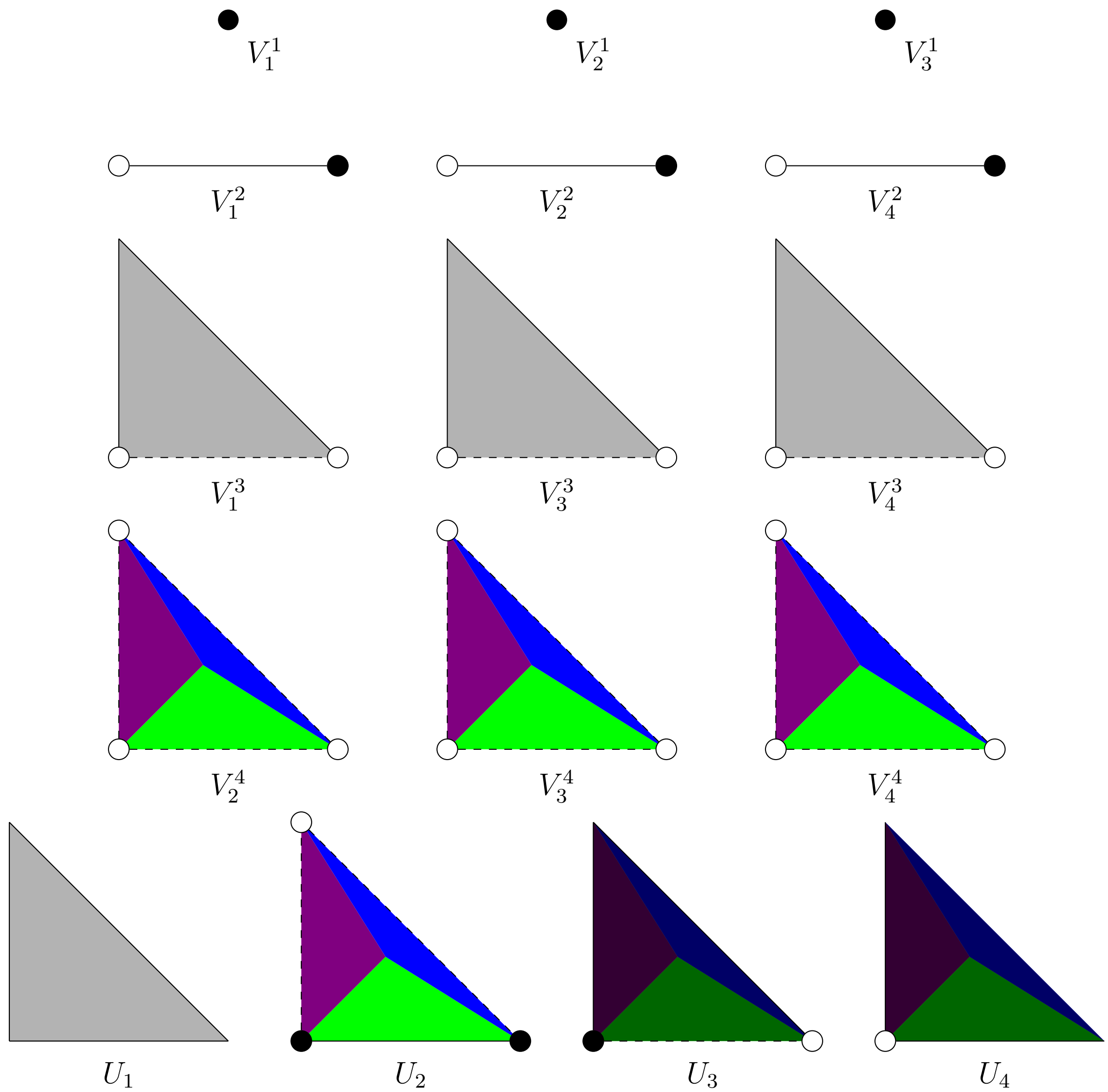
Figure 1: Constructing a convex realization of the code $\mathcal{C} = \{\emptyset, 12, 34, 123\}$, as in the proof of Theorem 1.



In the next example, we construct a realization of $\mathcal{C} = \{\emptyset, 123, 124, 134, 234\}$ in $\mathbb{R}^3$. A later result will prove that 3 is in fact the minimal embedding dimension of this code (Theorem 3).

Example 3. Consider the code $\mathcal{C} = \{\emptyset, 123, 124, 134, 234\}$. Figure 2 displays the convex sets U_1, U_2, U_3, and U_4 which realize $\mathcal{C}$ as well as the V^i_j that are used in the construction of the U_i's (see proof of in Theorem 1).

Figure 2: Constructing a convex realization of the code $\mathcal{C} = \{\emptyset, 123, 124, 134, 234\}$, as in the proof of Theorem 1.

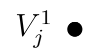


Theorem 1. *Every code is convex realizable. Moreover, the minimal embedding dimension of a code with m nonempty codewords is at most $m - 1$.*

Proof. We will give a construction for a convex realization of an arbitrary code.

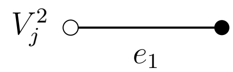
Let $\mathcal{C}$ be an arbitrary code on n neurons where $\mathcal{C} \setminus \{\emptyset\} = \{\sigma_1, \sigma_2, \ldots \sigma_k\}$. Let $\{e_1, ..., e_{k-1}\}$ be the standard basis for $\mathbb{R}^{k-1}$.

1. Take σ_1. Then for every $j \in [n]$, if $j \in \sigma_1$, define V_j^1 to be the closed point at the origin.



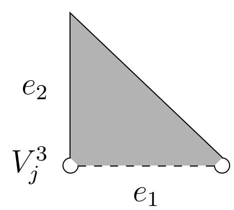
Otherwise, define $V_j^1 = \emptyset$.

2. Next take σ_2. Then for every $j \in [n]$, if $j \in \sigma_2$, define V_j^2 to be $\text{conv}\{0, e_1\} - \{0\}$.



Otherwise, define $V_j^2 = \emptyset$.

3. Next take σ_3. Then for every $j \in [n]$, if $j \in \sigma_3$, define V_j^3 to be $\text{conv}\{0, e_1, e_2\}$, but open along its intersection with $\text{conv}\{0, e_1\}$.



Otherwise, define $V_j^3 = \emptyset$.

4. Continuing in this way, for all $j \in [n]$, if $j \in \sigma_m$, define V_j^m to be $\text{conv}\{e_1, e_2, \ldots, e_{m-1}\}$, but open along its intersection with $\text{conv}\{0, e_1, e_2, \ldots e_{m-2}\}$. Otherwise, define $V_j^m = \emptyset$. Notice that by construction, V_j^m does not intersect any V_l^s constructed in a previous step where $s < j$.

5. When this has been completed for all $\sigma_j \in \mathcal{C}$, define

$$U_j = \bigcup_{i \in [k]} V_j^i$$

for all $j \in [n]$. We claim that $\mathcal{U} = \{U_1, U_2, \ldots, U_n\}$ is a convex realization of our code in $\mathbb{R}^{k-1}$. Note that for each $i \in [k]$, the codeword σ_i is realized by $\bigcup_{j \in \sigma_i} V_j^i$. Furthermore, for all $i \in [k]$, V_j^i are disjoint, so no additional codewords are realized. Thus $\mathcal{U}$ is a realization of $\mathcal{C}$. Furthermore, a face cannot affect the convexity of n-simplex unless the face itself is not convex. Similarly, a face cannot affect the convexity of a $n-1$-simplex unless the face itself is not convex. Thus, since each V_j^i is convex by construction, each U_j in our construction must be convex.

□

Next, we prove a result similar to that of Raffaella and Ngoc [5]. Rafaella and Ngoc showed that the minimal embedding dimension of an open connected code is at most three. Here, we give a sufficient condition for a convex open code to have a minimal embedding dimension of 2.

Definition 5. Let $\mathcal{C}$ be convex open and $\mathcal{U} = \{U_1, U_2, \ldots, U_n\}$ be a convex realization of $\mathcal{C}$. Then, smallest dimension d in which $\mathcal{C}$ is realizable by convex open U_i is its *minimal open embedding dimension*. Similarly, if $\mathcal{C}$ is convex closed and $\mathcal{U} = \{U_1, U_2, \ldots, U_n\}$ is a convex realization of $\mathcal{C}$, then smallest dimension d in which $\mathcal{C}$ is realizable by convex closed U_i is its *minimal closed embedding dimension*.

Theorem 2. *Suppose $\mathcal{C}$ is convex open and has a minimal open embedding dimension of 2. Then the minimal embedding dimension of $\mathcal{C}$ is 2.*

Proof. Note that it is equivalent to prove that, if $\mathcal{C}$ has a convex realization in dimension 1, then $\mathcal{C}$ has an open convex realization in dimension 1.

Let $\mathcal{C}$ be a neural code on n neurons with minimal embedding dimension $d = 1$ and let $\mathcal{U} = \{I_1, ..., I_n\}$ be a convex realization of $\mathcal{C}$ in dimension 1 where each I_k as an interval on the real number line. We will denote the left and right endpoints of an interval I_k by a_k and b_k respectively. Define

$$\varepsilon = \min(\{|a_i - a_j|, |b_i - b_j| \mid |a_i - a_j| > 0, |b_i - b_j| > 0\} \cup \{|a_i - b_j| \mid |a_i - b_j| > 0\})$$

In other words, ε is the smallest non zero distance between any two endpoints. Next, we will modify each interval in $\mathcal{U}$ to obtain a new set $\mathcal{U}'$ comprised of all open intervals which still realize $\mathcal{C}$. For every $I_k \in \mathcal{U}$, the following endpoint conditions of I_k give a construction for a modified interval, denoted I'_k:

- If $a_k \in I_k$, let $a_k - \varepsilon/3$ be the new, open endpoint.
- If $a_k \notin I_k$, let $a_k + \varepsilon/3$ be the new, open endpoint.
- If $b_k \in I_k$, let $a_k + \epsilon/3$ be the new, open endpoint.
- If $b_k \notin I_k$, let $a_k - \epsilon/3$ be the new, open endpoint.

Essentially, shrink I_k at open endpoints and extend I_k at closed enpoints. After completing this process for every $I_k \in \mathcal{U}$, let $\mathcal{U}' := \{I'_k = (a'_k, b'_k) \mid I_k \in \mathcal{U}\}$. Note that after our modification of the intervals, the distance between two endpoints can change by at most $2\epsilon/3$. By the construction of ϵ, the only possible points at which $\mathcal{U}'$ will have additional codewords or missing codewords as compared to $\mathcal{U}$ is where $\mathcal{U}$ had two intervals with equal endpoints, or a single interval with equal endpoints (ie. a single point).

Suppose $a_k = b_j, a_k \in I_k$ and $b_j \in I_j$. Then $I'_k \cap I'_j$ will be the interval (b_j, a_k), thus preserving the zone in $\mathcal{U}$ that had existed exactly at the point shared by a_k and b_j. All other cases follow similar logic. Thus $\mathcal{U}'$ is an open convex realization of $\mathcal{C}$ in dimension 1. Note that since in one dimension open convex and open connected sets are identical, this implies that any set that is convex realizable in 1 dimension is open connected realizable in 1 dimension. □

Our final main result, Theorem 3, will show that, in some cases, the dimension of the construction in Theorem 1 is the minimal embedding dimension of a code. Before this result, we provide a few definitions and two results, Lemma 1 and Lemma 2, which simplify the proof of Theorem 3.

Definition 6.

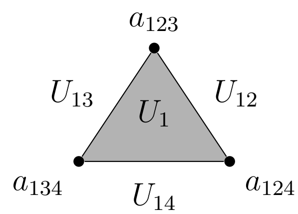
$$U_\sigma := \bigcap_{i\in\sigma} U_i$$

Definition 7. Let $\mathcal{C}_n$ be the code on n neurons containing exactly all of the codewords of length $n-1$,

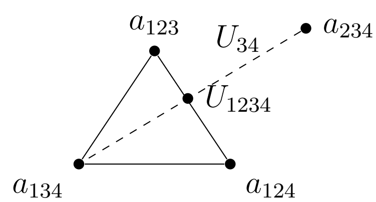
$$\mathcal{C}_n := \{\sigma \subseteq [n] \mid |\sigma| = n-1\}$$

Next, we will look more closely at the class of codes, $\mathcal{C}_n$ as defined above. Note that $|\mathcal{C}_n| = \binom{n}{n-1} = n$ for every n. In Theorem 3, we prove that for all n, $\mathcal{C}_n$ has minimal embedding dimension $n-1$, *thus showing that the construction in Theorem 1 cannot always be improved in terms of dimension.*

We begin with an example on 4 neurons: Let $\mathcal{C}_4 = \{123, 124, 134, 234\}$ and let $\mathcal{U} = \{U_1, U_2, U_3, U_4\}$ be a realization of $\mathcal{C}_4$. Then, there exist the following points: $a_{123} \in U_{123}$, $a_{124} \in U_{124}$, $a_{134} \in U_{134}$, and $a_{234} \in U_{234}$. Next, we will look at the convex hull of $\{a_{123}, a_{124}, a_{134}\}$. Note that by the convexity of each U_i, the edge between any a_σ and a_τ must be contained in $U_{\sigma\cap\tau}$. For example, the line segment between a_{123} and a_{124} must be contained in U_{12}. In the figure below, we label each edge with the U_σ containing that edge. Also, note that the entire convex hull must be contained in U_1.



Suppose for contradiction that a_{234} is coplanar with the other three a_σ. Note that a_{234} cannot intersect U_1 as that would imply that $1234 \in \mathcal{C}_4$. Thus a_{234} cannot be in $\mathrm{conv}\{a_{123}, a_{124}, a_{134}\}$. Placing a_{234} in an arbitrary location outside of $\mathrm{conv}\{a_{123}, a_{124}, a_{134}\}$ but still coplanar with the a_σ, we get the following:



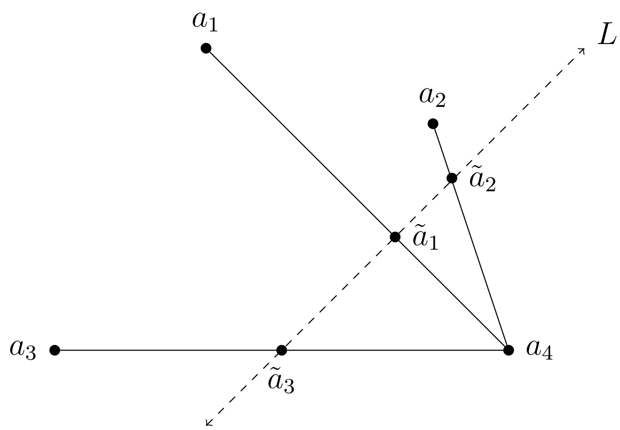
By convexity, the line between the points a_{134} and a_{234} must be contained in U_{34}. However, this line intersects the edge that is contained in U_{12}. Thus, the point of intersection must be contained in U_{1234}, resulting in a contradiction since $1234 \notin \mathcal{C}_4$. Note that such a contradiction occurs regardless of where the point a_{234} is placed. Our proof will generalize these ideas. First we introduce a definition and prove two supporting results.

Definition 8. A set of points $\{a_1, a_2, \dots a_n\}$ in $\mathbb{R}^{n-1}$ are *points in general linear position in* $\mathbb{R}^{n-1}$ if no hyperplane in $\mathbb{R}^{n-1}$ contains more than $n-1$ points.

Lemma 1. *Let $a_1, a_2, \ldots, a_n$ be points in $\mathbb{R}^{n-2}$ where $n \geq 4$. Let H be any hyperplane that separates conv$\{a_1, a_2, \ldots, a_{n-1}\}$ from a_n. Define $\tilde{a}_i$ for every $i \in [n-1]$ to be the intersection point of the line connecting a_i and a_n with H. If $a_1, a_2, \ldots, a_n$ are points in general linear position in $\mathbb{R}^{n-2}$, then $\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_{n-1}$ are points in general linear position in H.*

Proof. We will proceed by induction on n.

Base Case: Let $n = 4$. Let a_1, a_2, a_3, and a_4 be points in general linear position in $\mathbb{R}^2$. Consider a line L that separates conv$\{a_1, a_2, a_3\}$ from a_4. Then, define $\tilde{a}_i$ for $i \in [3]$ to be the intersection of the line connecting a_i and a_4 with L. An example of this projection is depicted below. Suppose for contradiction that $\tilde{a}_i = \tilde{a}_j$ for some $i, j \in [3]$, $i \neq j$. This implies that in $\mathbb{R}^2$, a_i, a_j, and a_4 are collinear, contradicting that they are points in general linear position in $\mathbb{R}^2$. Thus, $\tilde{a}_1, \tilde{a}_2$, and $\tilde{a}_3$ must be in general position in L.



Inductive Step: Assume that our claim holds for all $k < n$. Let $a_1, a_2, \ldots, a_n$ be points in general linear position in $\mathbb{R}^{n-2}$. Let H be any hyperplane that separates conv$\{a_1, a_2, \ldots, a_{n-1}\}$ from a_n. Define $\tilde{a}_i$ to be the intersection point of the line connecting a_i and a_n with H. Suppose for contradiction that $\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_{n-1}$ are not in general linear position in H.

Our inductive hypothesis implies that $\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_{n-2}$ are in general linear position in H. Thus, our contradiction must arise from a subset of at least $n-2$ of the $\tilde{a}_i$ that includes $\tilde{a}_{n-1}$. Call this set $\tilde{S}$. Then, all of the points in $\tilde{S}$ are coplanar in a plane of H. Define $S = \{a_i \mid \tilde{a}_i \in \tilde{S}\}$. Then, we get that a_n and all of the points in S lie in the same hyperplane of $\mathbb{R}^{n-2}$, contradicting that the points were in general linear position in $\mathbb{R}^{n-2}$. Thus, $\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_{n-1}$ are in general linear position in H. □

Lemma 2. *Assume $n \geq 3$ and let $a_1, a_2, \ldots, a_n$ be points in general linear position in $\mathbb{R}^{n-2}$. Assume that $a_n \notin$ conv$\{a_1, a_2, \ldots, a_n\}$. Then, there exists a partition $B_1 \cup B_2 = \{a_1, a_2, \ldots, a_{n-1}\}$ such that there exist points $t \in$ conv(B_1) and $z \in$ conv(B_2) with $t \neq z$, such that the three points, t, z, and a_n, are collinear.*

Proof. We will proceed by induction on n.

Base Case: Let $n = 3$. Then, a_1, a_2, and a_3 are points in general linear position in $\mathbb{R}^1$. Let $B_1 = \{a_1\}$ and $B_2 = \{a_2\}$. Then, conv$(B_1) = \{a_1\}$ and conv$(B_2) = \{a_2\}$. In $\mathbb{R}^1$, all points are collinear so if we let $t = a_1$ and $z = a_2$, we get that t, z, and a_3 are collinear. Since a_1, a_2, and a_3 are points in general linear position in $\mathbb{R}^1$, we know that $a_1 \neq a_2$ so $t \neq z$, completing our claim.

Inductive Step: Suppose our claim holds for all $k < n$.

Suppose $a_1, a_2, \dots, a_n$ are points in general linear position in $\mathbb{R}^{n-2}$. Let H be any hyperplane in $\mathbb{R}^{n-2}$ that separates $\text{conv}\{a_1, a_2, \dots, a_{n-1}\}$ from a_n. Then, define $\tilde{a_i}$ for all $i \in [n-1]$ to be the intersection point of the line connecting a_i and a_n with H. By Lemma 1, we know that $\tilde{a_1}, \tilde{a_2}, \dots, \tilde{a}_{n-1}$ are in general linear position in H. Then, by our inductive hypothesis, there exist sets $\tilde{B_1}, \tilde{B_2}$ which partition the set $\{\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_{n-2}\}$ such that there exist points $\tilde{t} \in \text{conv}(\tilde{B_1}), \tilde{z} \in \text{conv}(\tilde{B_2})$ and $\tilde{t} \neq \tilde{z}$ where $\tilde{t}, \tilde{z}$, and $\tilde{a}_{n-1}$ are collinear. Without loss of generality, assume that $\tilde{a}_{n-1}$ is closer to $\tilde{z}$ than $\tilde{t}$. Then, define the sets $B_1 = \{a_i \mid \tilde{a_i} \in \tilde{B_1}\} \cup \{a_{n-1}\}$ and $B_2 = \{a_i \mid \tilde{a_i} \in \tilde{B_2}\}$. We claim that B_1 and B_2 satisfy our claim.

First, note that $\tilde{z} \in \text{conv}(\tilde{B_1} \cup \{\tilde{a}_{n-1}\})$ by construction, since $\tilde{z} \in \text{conv}\{\tilde{t}, \tilde{a}_{n-1}\}$. Extend a line, L_z between a_n and $\tilde{z}$. Then, all points on L_z project to $\tilde{z}$ by our definition of this projection. Moreover, since $\tilde{z} \in \text{conv}(\tilde{B_1} \cup \{\tilde{a}_{n-1}\})$, L_z must intersect $\text{conv}(B_1)$, implying that there exists a point $z \in \text{conv}(B_1)$ such that the projection of z onto H is $\tilde{z}$. Similarly, since $\tilde{z} \in \text{conv}(\tilde{B_2})$, L_z must intersect $\text{conv}(B_2)$, implying that there exists a point $s \in \text{conv}(B_2)$ such that the projection of z onto H is $\tilde{z}$. Thus, s, z, and a_n all lie in L_z, meaning they are collinear. Lastly, since B_1 and B_2 form a partition of the $n-1$ points in $\mathbb{R}^2$, $\text{conv}(B_1)$ and $\text{conv}(B_2)$ are disjoint, giving us that $s \neq z$, thus proving our claim.

□

Theorem 3. *Let $\mathcal{C}_n$ be a code on n neurons as defined above. Then, the minimal embedding dimension of $\mathcal{C}_n$ is $n-1$. That is, the embedding dimension from Theorem 1 is exactly the minimal embedding dimension of $\mathcal{C}_n$ for every n.*

Proof. We will proceed by induction on the number of neurons.

Base Case: Let $n = 2$. Then, $\mathcal{C}_2 = \{1, 2\}$. Let $\mathcal{U} = \{U_1, U_2\}$ be a convex realization of $\mathcal{C}_2$. Note that U_1 and U_2 cannot have a point in common since $12 \notin \mathcal{C}_2$, so, by Theorem 1, $\mathcal{C}_2$ has minimal embedding dimension 1.

Inductive step: Assume for every $k < n$, the code $\mathcal{C}_k$ has minimal embedding dimension $k - 1$.

Write $\mathcal{C}_n = \{\sigma_1, \sigma_2, \dots \sigma_n\}$ where for each $i \in [n]$, $\sigma_i = [n] \backslash \{i\}$. Assume for contradiction that $\{U_1, U_2, \dots, U_n\}$ is a convex realization of $\mathcal{C}_n$ in $\mathbb{R}^{n-2}$. Then there exist a collection of points $a_1, a_2, \dots, a_n$ such that $a_i \in U_{\sigma_i}$ for all $i \in [n]$. Let $A = \{a_1, a_2, \dots, a_{n-1}\}$. We will begin by showing that $a_1, a_2, \dots, a_n$ are in general linear position in $\mathbb{R}^{n-2}$ and $a_n \notin \text{conv}\{a_1, a_2, \dots, a_{n-1}\}$ so that we can apply Lemma 2.

First, looking at $\sigma_1, \sigma_2, \sigma_3$, we can view these codewords as copies of the elements of $\mathcal{C}_3$. That is, we can view $2345 \dots n$, $1345 \dots n$, and $1245 \dots n$ as copies of the codewords $23, 13$, and 12. Then, by our inductive hypothesis, a_1, a_2, and a_3 cannot be collinear. In this way, for every $k < n$, any subset of k of our a_i cannot be contained in a $(k-2)$ dimensional plane of $\mathbb{R}^2$. This implies that $a_1, a_2, \dots, a_n$ must be in general linear position in $\mathbb{R}^{n-2}$.

Next, by our construction of the a_i, for every $i \in [n-1]$, $a_i \in U_n$. By convexity of U_n, we get that $\text{conv}(A) \subseteq U_n$. Then, looking at a_n, for every $j \in [n-1]$, $a_n \in U_j$. Thus, if $a_n \in \text{conv}(A)$, then $a_n \in U_n$, implying that the $123 \dots n \in \mathcal{C}_n$ which is a contradiction. Thus, $a_n \notin \text{conv}(A)$.

Applying Lemma 2, there exists a partition $B_1 = \{b_1, b_2, \dots, b_k\}$, $B_2 = \{b_{k+1}, b_{k+1}, \dots, b_{n-1}\}$ of $\{a_1, a_2, \dots, a_{n-1}\}$ and there exist points $t \in \text{conv}(B_1)$ and $z \in \text{conv}(B_2)$ such that t, z, and

a_n are collinear where a_n is closer to z.

Recall from above that for each $i \in [n]$, $\sigma_i = [n]\backslash i$ and $a_i \in U_{\sigma_i}$. Then, for each $b_i \in B_1$, there exists a codeword $\tau_i \in \mathcal{C}_n$ such that $b_i \in U_{\tau_i}$. Similarly, for each $b_j \in B_2$, there exists a codeword $\zeta_j \in \mathcal{C}_n$ such that $b_j \in U_{\zeta_j}$. Define

$$\tau = \bigcap_{i=1}^{k} \tau_i \qquad \zeta = \bigcap_{j=k+1}^{n-1} \zeta_j$$

Then, by the convexity of each U_i, we get that $\text{conv}(B_1) \subseteq U_\tau$ and $\text{conv}(B_2) \subseteq U_\zeta$. Since B_1 and B_2 partition A and σ_n is the only element of $\mathcal{C}_n$ not contained in A, we get that $\tau \cap \zeta = \{n\}$. By our constructions of τ and ζ, we have that $|\tau| + |\zeta| = n + 1$. However, since the intersection of τ and ζ is $\{n\}$, then $|\tau \cup \zeta| = n$, thus implying that $\tau \cup \zeta = [n]$.

To finish the proof, since $t \in \text{conv}(B_1) \subseteq U_\tau$, by convexity of the U_i, the line between t and a_n must be contained in $\bigcap_{i \in \tau \cap \sigma_n} U_i$. Since z is between t and a_n, then $z \in \bigcap_{i \in \tau \cap \sigma_n} U_i$. However, $z \in \text{conv}(B_2) \subseteq U_\zeta$ so $z \in U_\zeta \cap \left(\bigcap_{i \in \tau \cap \sigma_n} U_i \right)$. This gives us that the codeword $\zeta \cup (\tau \cap \sigma_n)$ is realized at z. By our constructions of τ and ζ, we get that $\zeta \cup (\tau \cap \sigma_n) = [n]$, so the codeword $123 \ldots n$ is realized at z, contradicting our assumption that the codeword $123 \ldots n \notin \mathcal{C}_n$. Thus, $\mathcal{C}_n$ is not convex realizable in $\mathbb{R}^{n-2}$. By Theorem 1, $\mathcal{C}_n$ has a realization in dimension $n - 1$. Thus, the minimal embedding dimension of $\mathcal{C}_n$ is $n - 1$.

□

Corollary 1. *The minimal embedding dimension of all neural codes has no upper bound.*

Proof. This follows immediately from Theorem 3. □

3 Discussion

We have proven by construction that every code $\mathcal{C}$ has a convex realization in $\mathbb{R}^{d-1}$ where d is the number of nonempty codewords in $\mathcal{C}$. While the application of this construction in place cells might not be realistic because of the potential for high dimension, a few related questions follow naturally which could lead to more insight into the behavior of place cells. Since we have shown that every code is convex realizable, can we determine the minimal embedding dimension? If a code is convex open or closed, when is the minimal open or closed embedding dimension strictly greater than the minimal embedding dimension?

Our results in Theorems 2 and 3 provide some framework for answering these questions. Theorem 2 provides sufficient conditions for the minimal open embedding dimension of a code to be equal to the minimal embedding dimension in dimensions 1 and 2. Theorem 3 implies that, in certain cases, the dimension of the construction in Theorem 1 is exactly the minimal embedding dimension. Moreover, Theorem 3 implies that there is no upper bound on the minimal embedding dimensions of all codes.

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