Convex Codes and Minimal Embedding Dimensions

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- Each place cell corresponds to a receptive field
- The receptive fields from a set of neurons give us a neural code

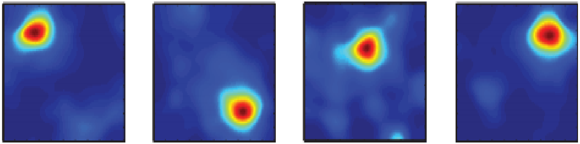
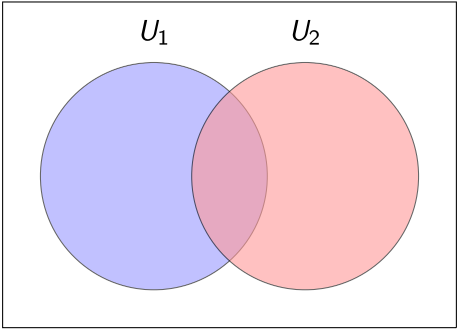


Figure: Place Cells

Neural Code Example

Convex Code: $\{\emptyset, 1, 2, 12\}$



Convex Neural Codes

Definition

We say that a code $\mathcal{C}$ is a *convex code on n neurons* if there exists a collection of sets $\mathcal{U} = \{U_1, U_2, \dots, U_n\}$ such that for each $i \in [n]$, U_i is a convex subset of $\mathbb{R}^d$ and $\mathcal{C}(\mathcal{U}) = \mathcal{C}$. A code $\mathcal{C} = \mathcal{C}(\mathcal{U})$ is *open convex* or *closed convex* if the $U_i \in \mathcal{U}$ are all open or all closed.

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Goal

Classify which codes are convex open, convex closed, just convex, or not convex at all.

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Classify which codes are convex open, convex closed, just convex, or not convex at all.

Theorem (F., Muthiah)

Every neural code is just convex.

Definitions

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Let $X_1, X_2, \ldots, X_n$ be subsets of $\mathbb{R}^d$. Define the *convex hull* of $X_1, X_2, \ldots, X_n$ to be the smallest convex set in $\mathbb{R}^d$ containing $X_1, X_2, \ldots, X_n$, denoted by $\operatorname{conv}(X_1, X_2, \ldots, X_n)$.

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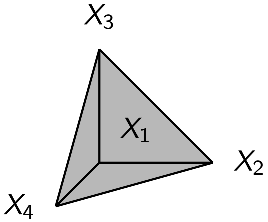
Let $X_1 = (0,0,0)$, $X_2 = (1,0,0)$, $X_3 = (0,1,0)$, and $X_4 = (0,0,1)$. Then the convex hull of $\{X_1, X_2, X_3, X_4\}$ is



Just Convex Construction

Let $\mathcal{C}$ be a code on n neurons where $\mathcal{C} \setminus \{\emptyset\} = \{\sigma_1, \sigma_2, \ldots, \sigma_k\}$ and let $\{e_1, ..., e_{k-1}\}$ be the standard basis for $\mathbb{R}^{k-1}$.

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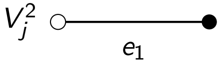
Take σ_1. Then for every $j \in [n]$, if $j \in \sigma_1$ define V_j^1 to be the closed point at the origin.

$$V_j^1 \bullet$$

Otherwise, define $V_j^1 = \emptyset$.

Just Convex Construction

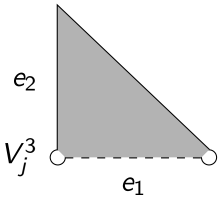
Take σ_2. Then for every $j \in [n]$, if $j \in \sigma_2$ define V_j^2 to be $\text{conv}\{0, e_1\} - \{0\}$.



Otherwise, define $V_j^2 = \emptyset$.

Just Convex Construction

Next take σ_3. Then for every $j \in [n]$, if $j \in \sigma_3$ define V_j^3 to be $\text{conv}\{0, e_1, e_2\}$, but open along its intersection with $\text{conv}\{0, e_1\}$.



Otherwise, define $V_j^3 = \emptyset$.

Just Convex Construction

Continuing in this way, for all $j \in [n]$, if $j \in \sigma_m$, define V_j^m to be $\text{conv}\{0, e_1, e_2, \ldots, e_{m-1}\}$, but open along its intersection with $\text{conv}\{0, e_1, e_2, \ldots, e_{m-2}\}$. Otherwise, define $V_j^m = \emptyset$.

Just Convex Construction

When this has been completed for all $\sigma_j \in \mathcal{C}$, define

$$U_j = \bigcup_{i \in [k]} V_j^i = V_j^1 \cup V_j^2 \cup \ldots \cup V_j^k$$

for all $j \in [n]$.

Example

Let $\mathcal{C} = \{\emptyset, 12, 13, 23\}$.

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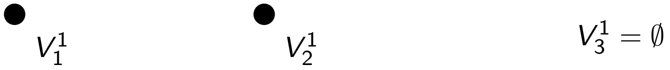

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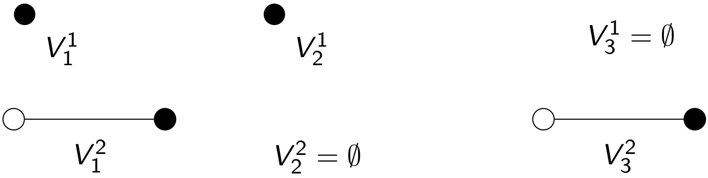

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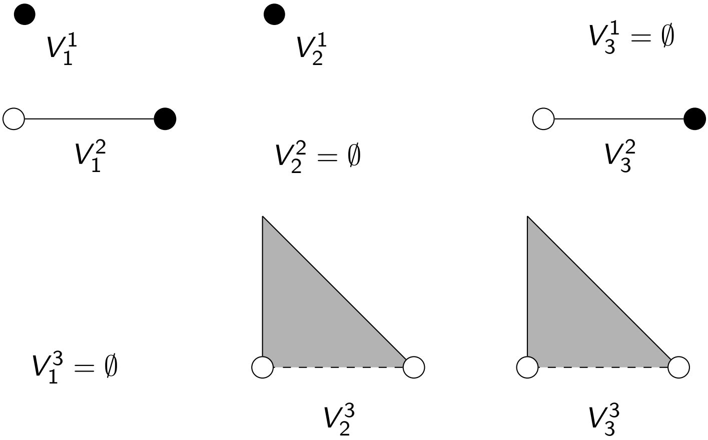

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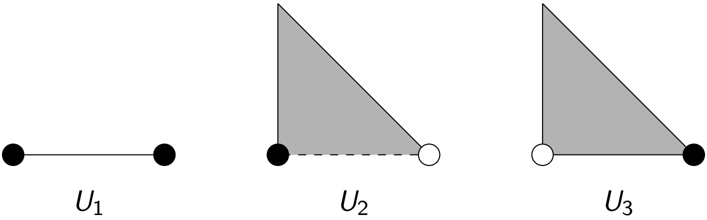

Minimal Embedding Dimension

$\{\emptyset, 1, 2, 3, 4, 5, 12, 15, 23, 24, 25, 34, 45, 56, 125, 234, 245\}$

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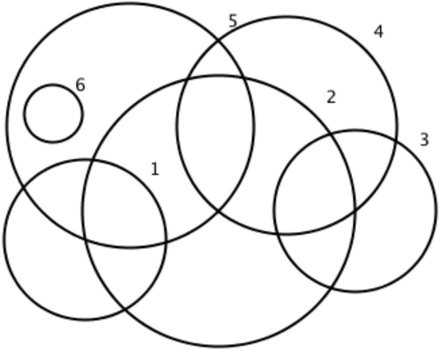

Minimal Embedding Dimension

Definition

Let $\mathcal{C}$ be a convex code on n neurons. Suppose $\mathcal{C}$ is realized by $\mathcal{U} = \{U_1, U_2, \ldots, U_n\}$ where each $U_i \subset \mathbb{R}^d$ is convex.

- The minimal such d is the *minimal embedding dimension* of $\mathcal{C}$.
- If we require all $U_i \in \mathcal{U}$ to be open, the minimal such d is the *minimal open embedding dimension* of $\mathcal{C}$.
- If we require all $U_i \in \mathcal{U}$ to be closed, the minimal such d is the *minimal closed embedding dimension* of $\mathcal{C}$.

Minimal Embedding Dimension

Definition

Define $\mathcal{C}_n$ to be the code on n neurons containing all codewords of length $n-1$,

$$\mathcal{C}_n = \{\sigma \mid \sigma \subseteq [n], |\sigma| = n-1\}.$$

Note that $|\mathcal{C}_n| = \binom{n}{n-1} = n$.

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Theorem (F., Muthiah)

For every n, $\mathcal{C}_n$ has minimal embedding dimension $n-1$.

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Suppose toward contradiction that $\mathcal{C}_3$ has a realization in 1 dimension.

Then, a_{12}, a_{13}, and a_{23} must be collinear.

Example

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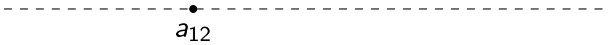

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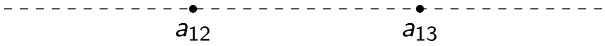

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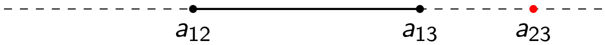

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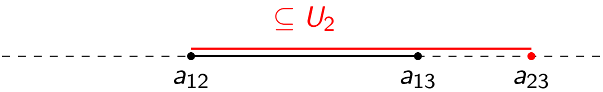

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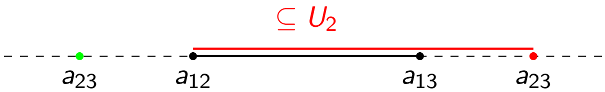

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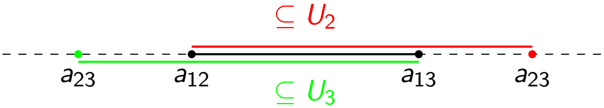

Discussion

New Questions:

- Since every code is convex, what is the minimal embedding dimension of an arbitrary code?
- When is the minimal open/closed embedding dimension strictly greater than the minimal embedding dimension of a code?
- When is the minimal open/closed embedding dimension equal to the minimal embedding dimension of a code?

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