Patterns arising in the Kernel of Generalized Dedekind Sums (Part 1)

Evuilynn Nguyen
Rhodes College

Juan Ramirez
University of Houston

July 20, 2020

Overview

- Background
  - Introduction
  - Dirichlet Characters
- Dedekind Sum
  - Classical Dedekind Sum
  - Generalized Dedekind Sums
- Results
  - Previous Results
  - Our Results

What Is The Dedekind Sum?

Consider the map $S_{\chi_1,\chi_2} : \Gamma_0(q_1 q_2) \to \mathbb{C}$, by:

$$S_{\chi_1,\chi_2}(\gamma) = \sum_{j \pmod{c}} \sum_{n \pmod{q_1}} \overline{\chi_2}(j)\, \overline{\chi_1}(n)\, B_1\left(\frac{j}{c}\right) B_1\left(\frac{n}{q_1} + \frac{aj}{c}\right)$$

Other Areas:

1. Combinatorial Geometry
2. Algebraic Number Theory

Dirichlet Characters

A **Dirichlet character** modulo q is a function $\chi : \mathbb{Z} \to \mathbb{C}$ with the following properties:

Dirichlet Characters

A **Dirichlet character** modulo q is a function $\chi : \mathbb{Z} \to \mathbb{C}$ with the following properties:

1. $\chi(n + lq) = \chi(n), \forall n, l \in \mathbb{Z}$

Dirichlet Characters

A **Dirichlet character** modulo q is a function $\chi : \mathbb{Z} \to \mathbb{C}$ with the following properties:

1. $\chi(n + lq) = \chi(n), \forall n, l \in \mathbb{Z}$ (periodic, period q)

Dirichlet Characters

A **Dirichlet character** modulo q is a function $\chi : \mathbb{Z} \to \mathbb{C}$ with the following properties:

1. $\chi(n + lq) = \chi(n)$, $\forall n, l \in \mathbb{Z}$ (periodic, period q)
2. $\chi(mn) = \chi(m)\chi(n)$

Dirichlet Characters

A **Dirichlet character** modulo q is a function $\chi : \mathbb{Z} \to \mathbb{C}$ with the following properties:

1. $\chi(n + lq) = \chi(n), \forall n, l \in \mathbb{Z}$ (periodic, period q)
2. $\chi(mn) = \chi(m)\chi(n)$
3. $\chi(n) = 0$ if $\gcd(n, q) > 1$ and $\chi(n) \neq 0$ if $\gcd(n, q) = 1$

Dirichlet Characters

A **Dirichlet character** modulo q is a function $\chi : \mathbb{Z} \to \mathbb{C}$ with the following properties:

1. $\chi(n + lq) = \chi(n), \forall n, l \in \mathbb{Z}$ (periodic, period q)
2. $\chi(mn) = \chi(m)\chi(n)$
3. $\chi(n) = 0$ if $\gcd(n, q) > 1$ and $\chi(n) \neq 0$ if $\gcd(n, q) = 1$
4. $\chi(1) = 1$

Dirichlet Characters

A **Dirichlet character** modulo q is a function $\chi : \mathbb{Z} \to \mathbb{C}$ with the following properties:

1. $\chi(n + lq) = \chi(n), \forall n, l \in \mathbb{Z}$ (periodic, period q)
2. $\chi(mn) = \chi(m)\chi(n)$
3. $\chi(n) = 0$ if $\gcd(n, q) > 1$ and $\chi(n) \neq 0$ if $\gcd(n, q) = 1$
4. $\chi(1) = 1$

Ex:

The **Legendre symbol**: $\left(\frac{a}{q}\right) \equiv a^{\frac{q-1}{2}} \pmod q$

Dirichlet Characters: Example

Take the **Legendre symbol**: $\left(\frac{a}{q}\right) \equiv a^{\frac{q-1}{2}} \pmod{q}$ for an odd prime q.

Dirichlet Character Properties

1. $\chi(n+lq) = \chi(n), \forall n, l \in \mathbb{Z}$
2. $\chi(mn) = \chi(m)\chi(n)$
3. $\chi(n) = 0$ if $\gcd(n,q) > 1$ and $\chi(n) \neq 0$ if $\gcd(n,q) = 1$
4. $\chi(1) = 1$

Dirichlet Characters: Example

Take the **Legendre symbol**: $\left(\frac{a}{q}\right) \equiv a^{\frac{q-1}{2}} \pmod q$ for an odd prime q.

Dirichlet Character Properties

1. $\chi(n+lq) = \chi(n), \forall n, l \in \mathbb{Z}$
2. $\chi(mn) = \chi(m)\chi(n)$
3. $\chi(n) = 0$ if $\gcd(n,q) > 1$ and $\chi(n) \neq 0$ if $\gcd(n,q) = 1$
4. $\chi(1) = 1$

1. $\left(\frac{n+lq}{q}\right) \equiv (n+lq)^{\frac{q-1}{2}} \equiv n^{\frac{q-1}{2}} \equiv \left(\frac{n}{q}\right) \pmod q$

Dirichlet Characters: Example

Take the **Legendre symbol**: $\left(\frac{a}{q}\right) \equiv a^{\frac{q-1}{2}} \pmod{q}$ for an odd prime q.

Dirichlet Character Properties

1. $\chi(n + lq) = \chi(n), \forall n, l \in \mathbb{Z}$
2. $\chi(mn) = \chi(m)\chi(n)$
3. $\chi(n) = 0$ if $\gcd(n, q) > 1$ and $\chi(n) \neq 0$ if $\gcd(n, q) = 1$
4. $\chi(1) = 1$

1. $\left(\frac{n+lq}{q}\right) \equiv (n + lq)^{\frac{q-1}{2}} \equiv n^{\frac{q-1}{2}} \equiv \left(\frac{n}{q}\right) \pmod{q}$
2. $\left(\frac{mn}{q}\right) \equiv (mn)^{\frac{q-1}{2}} \equiv \left(m^{\frac{q-1}{2}}\right)\left(n^{\frac{q-1}{2}}\right) \equiv \left(\frac{m}{q}\right)\left(\frac{n}{q}\right) \pmod{q}$

Dirichlet Characters: Example

Take the **Legendre symbol**: $\left(\frac{a}{q}\right) \equiv a^{\frac{q-1}{2}} \pmod{q}$ for an odd prime q.

Dirichlet Character Properties

1. $\chi(n + lq) = \chi(n), \forall n, l \in \mathbb{Z}$
2. $\chi(mn) = \chi(m)\chi(n)$
3. $\chi(n) = 0$ if $\gcd(n, q) > 1$ and $\chi(n) \neq 0$ if $\gcd(n, q) = 1$
4. $\chi(1) = 1$

1. $\left(\frac{n+lq}{q}\right) \equiv (n + lq)^{\frac{q-1}{2}} \equiv n^{\frac{q-1}{2}} \equiv \left(\frac{n}{q}\right) \pmod{q}$
2. $\left(\frac{mn}{q}\right) \equiv (mn)^{\frac{q-1}{2}} \equiv \left(m^{\frac{q-1}{2}}\right)\left(n^{\frac{q-1}{2}}\right) \equiv \left(\frac{m}{q}\right)\left(\frac{n}{q}\right) \pmod{q}$
3. $\left(\frac{n}{q}\right) \equiv 0 \pmod{q}$ iff $n \equiv 0 \pmod{q}$

Dirichlet Characters: Example

Take the **Legendre symbol**: $\left(\frac{a}{q}\right) \equiv a^{\frac{q-1}{2}} \pmod{q}$ for an odd prime q.

Dirichlet Character Properties

1. $\chi(n + lq) = \chi(n), \forall n, l \in \mathbb{Z}$
2. $\chi(mn) = \chi(m)\chi(n)$
3. $\chi(n) = 0$ if $\gcd(n, q) > 1$ and $\chi(n) \neq 0$ if $\gcd(n, q) = 1$
4. $\chi(1) = 1$

1. $\left(\frac{n+lq}{q}\right) \equiv (n+lq)^{\frac{q-1}{2}} \equiv n^{\frac{q-1}{2}} \equiv \left(\frac{n}{q}\right) \pmod{q}$
2. $\left(\frac{mn}{q}\right) \equiv (mn)^{\frac{q-1}{2}} \equiv \left(m^{\frac{q-1}{2}}\right)\left(n^{\frac{q-1}{2}}\right) \equiv \left(\frac{m}{q}\right)\left(\frac{n}{q}\right) \pmod{q}$
3. $\left(\frac{n}{q}\right) \equiv 0 \pmod{q}$ iff $n \equiv 0 \pmod{q}$
4. $\left(\frac{1}{q}\right) \equiv 1^{\frac{q-1}{2}} \equiv 1 \pmod{q}$

Primitive Characters

The *trivial character* $\chi_0 \pmod{q}$ is defined as

$$\chi_0(n) = \begin{cases} 1 & \gcd(n, q) = 1 \\ 0 & \text{otherwise} \end{cases}$$

A character $\chi \pmod{ql}$ is **primitive** if it cannot be written as

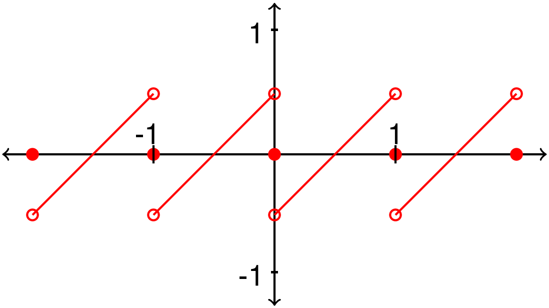
$$\chi = \chi^* \chi_0$$

for some $\chi^* \pmod{l}$.

Classical Dedekind Sum

Bernoulli Function

$$B_1(x) = \begin{cases} x - \lfloor x \rfloor - \frac{1}{2} & x \in \mathbb{R} \setminus \mathbb{Z} \\ 0 & x \in \mathbb{Z} \end{cases}$$



Classical Dedekind Sums

Definition:

Let a and c be co-prime integers with $c > 0$. Then we define the **classical dedekind sum** as

$$s(a,c) = \sum_{j \pmod{c}} B_1\left(\frac{j}{c}\right) B_1\left(\frac{aj}{c}\right)$$

where B_1 is the Bernoulli function.

Reciprocity Law:

$$s(a,c) + s(c,a) = \frac{1}{12}\left(\frac{a}{c} + \frac{1}{ac} + \frac{c}{a}\right) - \frac{1}{4}$$

Generalized Dedekind Sums

$SL_2(\mathbb{Z}), \Gamma_0(N), \Gamma_1(N), \& \ \Gamma(N)$:

$$SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, \text{ and det} = 1 \right\}$$

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : a \equiv d \equiv 1 \mod N, \text{ and } c \equiv 0 \pmod{N} \right\}$$

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : a \equiv d \equiv 1 \mod N, \text{ and } b \equiv c \equiv 0 \pmod{N} \right\}$$

$$\Gamma(N) \subset \Gamma_1(N) \subset \Gamma_0(N) \subset SL_2(\mathbb{Z})$$

Generalized Dedekind Sums

$\Gamma_0(N)$

$$\Gamma_0(q_1q_2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 (\bmod q_1q_2) \right\}$$

Let χ_1, χ_2 be non-trivial primitive Dirichlet characters modulo q_1 and q_2, respectively, such that $q_1, q_2 > 1$. The **generalized Dedekind sum** is

$$S_{\chi_1,\chi_2}(\gamma) = S_{\chi_1,\chi_2}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \sum_{j \pmod c} \sum_{n \pmod{q_1}} \overline{\chi_2}(j)\, \overline{\chi_1}(n)\, B_1\left(\frac{j}{c}\right) B_1\left(\frac{n}{q_1} + \frac{aj}{c}\right)$$

where $\gamma \in \Gamma_0(q_1q_2)$.

Generalized Dedekind Sums

$\Gamma_0(N)$

$$\Gamma_0(q_1q_2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 (\mathrm{mod} q_1q_2) \right\}$$

Let χ_1, χ_2 be non-trivial primitive Dirichlet characters modulo q_1 and q_2, respectively, such that $q_1, q_2 > 1$. The **generalized Dedekind sum** is

$$S_{\chi_1,\chi_2}(\gamma) = S_{\chi_1,\chi_2}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \sum_{j \pmod{c}} \sum_{n \pmod{q_1}} \overline{\chi_2}(j)\, \overline{\chi_1}(n)\, B_1\left(\frac{j}{c}\right) B_1\left(\frac{n}{q_1} + \frac{aj}{c}\right)$$

where $\gamma \in \Gamma_0(q_1q_2)$.

$$S_{\chi_1,\chi_2}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = S_{\chi_1,\chi_2}(a, c).$$

Generalized Dedekind Sums

Reciprocity Formula: Theorem (Stucker-Vennos-Young, 2018)

For $\gamma \in \Gamma_0(q_1q_2)$, let $\gamma' = \begin{pmatrix} d & -c \\ -bq_1q_2 & a \end{pmatrix} \in \Gamma_0(q_1q_2)$. Then if χ_1, χ_2 are even, then

$$S_{\chi_1,\chi_2}(\gamma) = S_{\chi_2,\chi_1}(\gamma').$$

If χ_1, χ_2 are odd, then

$$S_{\chi_1,\chi_2}(\gamma) = -S_{\chi_2,\chi_1}(\gamma') + (1 - \psi(\gamma))\left(\frac{\tau(\overline{\chi}_1)\tau(\overline{\chi}_2)}{(\pi i)^2}\right)L(1,\chi_1)L(1,\chi_2).$$

Generalized Dedekind Sums

Reciprocity Formula: Theorem (Stucker-Vennos-Young, 2018)

For $\gamma \in \Gamma_0(q_1 q_2)$, let $\gamma' = \begin{pmatrix} d & -c \\ -bq_1q_2 & a \end{pmatrix} \in \Gamma_0(q_1q_2)$. Then if χ_1, χ_2 are even, then

$$S_{\chi_1,\chi_2}(\gamma) = S_{\chi_2,\chi_1}(\gamma').$$

If χ_1, χ_2 are odd, then

$$S_{\chi_1,\chi_2}(\gamma) = -S_{\chi_2,\chi_1}(\gamma') + (1 - \psi(\gamma))\left(\frac{\tau(\overline{\chi}_1)\tau(\overline{\chi}_2)}{(\pi i)^2}\right)L(1,\chi_1)L(1,\chi_2).$$

Reciprocity Law:

$$s(a,c) + s(c,a) = \frac{1}{12}\left(\frac{a}{c} + \frac{1}{ac} + \frac{c}{a}\right) - \frac{1}{4}$$

Motivation

Lemma (Stucker-Vennos-Young, 2018)

Let $\gamma_1, \gamma_2 \in \Gamma_0(q_1 q_2)$. Then

$$S_{\chi_1,\chi_2}(\gamma_1\gamma_2) = S_{\chi_1,\chi_2}(\gamma_1) + \psi(\gamma_1) S_{\chi_1,\chi_2}(\gamma_2).$$

Motivation

Lemma (Stucker-Vennos-Young, 2018)

Let $\gamma_1, \gamma_2 \in \Gamma_0(q_1 q_2)$. Then

$$S_{\chi_1,\chi_2}(\gamma_1\gamma_2) = S_{\chi_1,\chi_2}(\gamma_1) + \psi(\gamma_1)S_{\chi_1,\chi_2}(\gamma_2).$$

Homomorphism:

For $\gamma \in \Gamma_1(q_1 q_2)$, we get $\psi(\gamma) = 1$. Therefore,

$$S_{\chi_1,\chi_2}(\gamma_1\gamma_2) = S_{\chi_1,\chi_2}(\gamma_1) + S_{\chi_1,\chi_2}(\gamma_2).$$

Motivation

Lemma (Stucker-Vennos-Young, 2018)

Let $\gamma_1, \gamma_2 \in \Gamma_0(q_1 q_2)$. Then

$$S_{\chi_1,\chi_2}(\gamma_1\gamma_2) = S_{\chi_1,\chi_2}(\gamma_1) + \psi(\gamma_1) S_{\chi_1,\chi_2}(\gamma_2).$$

Homomorphism:

For $\gamma \in \Gamma_1(q_1 q_2)$, we get $\psi(\gamma) = 1$. Therefore,

$$S_{\chi_1,\chi_2}(\gamma_1\gamma_2) = S_{\chi_1,\chi_2}(\gamma_1) + S_{\chi_1,\chi_2}(\gamma_2).$$

Research Question

What is the kernel of the S_{χ_1,χ_2}, i.e, for what $\gamma \in \Gamma_0(q_1 q_2)$ is $S_{\chi_1,\chi_2}(\gamma) = 0$?

Motivation

The First Isomorphism Theorem

If $\varphi : G \to H$ is a homormorphism of groups, then $G/ker(\varphi) \cong im(\varphi)$

Motivation

The First Isomorphism Theorem

If $\varphi : G \to H$ is a homormorphism of groups, then $G/ker(\varphi) \cong im(\varphi)$

Our Case?

If $S_{\chi_1,\chi_2} : \Gamma_1(q_1q_2) \to \mathbb{C}$ is a homomorphism of groups, then

$$\Gamma_1(q_1q_2)/ker(S_{\chi_1,\chi_2}) \cong S_{\chi_1,\chi_2}(\Gamma_1(q_1q_2))$$

Previous Work

Theorem (Dillon-Gaston, 2019)

Let χ_1 and χ_2 be nontrivial primitive characters modulo q_1 and q_2, respectively, such that $\chi_1\chi_2(-1) = 1$. Then for each positive $c \equiv 0 \mod q_1q_2$, there exists an integer a coprime to c such that $S_{\chi_1,\chi_2}(a, c)$ is nonzero.

Commutator Subgroup

Definition:

The commutator subgroup C of group G is the subgroup generated by $\{ghg^{-1}h^{-1} : \forall g, h \in G\}$.

Fact:

Let C_1 denote the commutator subgroup of $\Gamma_1(q_1q_2)$ and $\gamma \in \Gamma_1(q_1q_2)$. Since $S_{\chi_1,\chi_2} : \Gamma_1(q_1q_2) \to \mathbb{C}$ is a group homomorphism, and $(\mathbb{C}, +)$ is abelian, then

$$C_1 \subset ker(S_{\chi_1,\chi_2}(\Gamma_1(q_1q_2)))$$

References

[1] T. Apostol, *Modular Functions and Dirichlet Series in Number Theory.* Graduate Text in Mathematics, 41. Springer-Verlag,

[2] T. Apostol, *Introduction to Analytic Number Theory.* Undergraduate Text in Mathematics

[3] T. Dillon, S. Gaston, *An average of generalized Dedekind sums.* Journal of Number Theory, 212, (2020) 323-338.

[4] F. Diamond, J. Shurman, *A First Course in Modular Forms.* Graduate Text in Mathematics, 228. Springer-Verlag, New York, 2005.

[5] T. Stucker, A. Vennos, M. P. Young, *Dedekind sums arising from newform Eisenstein series.* preprint

Thank You!

We would like to thank Dr. Young (Mentor), Agniva Dasgupta (TA), the organizers and TA's of the REU, Texas A&M, and NSF!