Texas Geometry and Topology Conference

This is a report on the presentations at the 31th meeting of the Texas Geometry and Topology Conference at Texas Christian University, February 27-29, 2004. This conference was partially supported by National Science Foundation Grant DMS-0306628 and Texas Christian University. Speakers reported on recent research. For this report, speakers have provided synopses of their talks together with broader discussions of the significance and context of their results.

Meeting 31. Texas Christian University, February 27-29, 2004

Cameron Gordon, The University of Texas at Austin, *Knots with unknotting number* 1

The *unknotting number* $u(K)$ of a knot K is the least number of times that K must be allowed to pass through itself in order to transform it into the unknot. It is one of the oldest invariants in knot theory, going back to P. G. Tait in the 1870's, who called it the *beknottedness* of K. In the talk, we surveyed some of the results that have been used to show that a knot has unknotting number greater than 1, with specific reference to the problem of determining the knots K with $u(K) = 1$ among those with crossing number $c(K) \leq 9$ (and, ultimately, $c(K) \leq 10$).

There are 85 (prime) knots with $c(K) \leq 9$ (including the unknot), of which 35 are easily seen to have $u(K) = 1$. The goal is to prove that the remaining 49 non-trivial knots indeed have $u(K) > 1$. The first invariant to be used in this context was the *signature* $\sigma(K) \in 2\mathbb{Z}$. It is clear that a crossing-change can change $\sigma(K)$ by at most 2; hence $|\sigma(K)| \geq 4$ implies $u(K) > 1$. This condition sows that 25 of the 49 knots have $u(K) > 1$.

All the other results that we discussed were based on the *Montesinos correspondence* [M]: if $u(K) = 1$, then $B_2(K)$, the 2-fold branched cover of K, is of the form $K^*(m/2)$; i.e., can be obtained by $m/2$-Dehn surgery on some knot K^* in S^3. It follows immediately that if $u(K) = 1$, then $H_1(B_2(K))$ must be cyclic, and this rules out an additional 8 knots with $c(K) \leq 9$. Lickorish [L] observed that the Montesinos correspondence also gives a restriction on the linking form $\lambda : H_1(B_2(K)) \times H_1(B_2(K)) \to \mathbb{Q}/\mathbb{Z}$: $H_1(B_2(K))$ must have a generator x such that $\lambda(x, x) = 2/m \in \mathbb{Q}/\mathbb{Z}$. This rules out a further 6 knots.

If K is a 2-bridge knot, then $B_2(K)$ is a lens space, while the Cyclic Surgery Theory [CGLS] says that if $K^*(m/\ell)$, $\ell > 1$, has cyclic fundamental group, then K^* is a torus knot. Using this, Kanenobu and Murakami [KM] determined exactly which 2-bridge knots have $u(K) = 1$. Applying this leaves 4 knots: 8_{10}, 9_{25}, 9_{29}, 9_{32}. Kobayashi [K] showed that $u((_{25}) > 1$, by proving that if $u(K) = 1$, then the unknotting crossing-change must come from untwisting a plumbed Hopf band on some minimal genus Seifert surface for K.

A knot K is *double composite* if there is a 2-sphere S in S^3 which meets K in four points and decomposes (S^3, K) into two non-split tangles (B_i^3, τ_i), $i = 1, 2$. Scharlemann [S] showed that a composite knot $K = K_1 \# K_2$ has $u(K) > 1$; (this also follows from using the Montesinos correspondence, from the $\mathbb{Z}_2$-Smith Conjecture and the fact that if $K^*(m/\ell)$ is reducible, then $\ell = 1$ [GL1]). By contrast, it is easy to construct doubly composite knots with unknotting number 1, if we allow the *unknotting arc* that determines the crossing-change to be disjoint from the decomposing sphere S. It is much harder to find examples where the unkotting arc must meet S, but an explicit infinite family $\mathcal{E}$ of such knots was constructed by Eudave-Muñoz [E]. Now if K is doubly composite, then $B_2(K) = B_2(\tau_1) \cup_T B_2(\tau_2)$, where $B_2(\tau_i)$ is the 2-fold branched cover of the tangle τ_i, and since τ_i is non-split, the torus T is incompressible in $B_2(\tau_i)$, $i = 1, 2$. Thus $B_2(K)$ is *toroidal*; i.e., contains an imcompressible torus. Hence, by the Montesinos correspondence, we get a family $\mathcal{E}^*$ of hyperbolic knots K^* in S^3, each with a toroidal Dehn surgery $K^*(m/2)$. Now in [GL2] it is shown that these are the only hyperbolic knots with non-integral toroidal Dehn surgeries. This leads to a strong condition for a doubly composite knot to have $u(K) = 1$, a special case of which is

Theorem. *Let K be a doubly composite knot with a unique decomposing sphere K, and a suppose $u(K) = 1$. Then either $K \in \mathcal{E}$ or any unknotting arc for K can be made disjoint from S.*

Using this theorem, one can easily show that $u(9_{29})$ and $u(9_{32})$ are > 1. This leaves only 8_{10} undecided.

This was the situation up until very recently. Now, however, Ozvath and Szabo [OS] have used their Heegaard Floer homology theory to show that $u(8_{10}) > 1$. Their methods apply in principle to all the alternating knots (for example, they also show that $u(9_{29})$ and $u(9_{32})$ are > 1), and to other knots as well. In fact, using the signature condition, the cyclicity of $H_1(B_2(K))$, and the Ozvath-Szabo theorem, the knots with unknotting number 1 and $c(K) \leq 10$ are determined, with the exception of 10_{145} and 10_{153}. Tanaka has shown [T] that $u(10_{145}) > 1$, and since 10_{153} is doubly composite, one can show that $u(10_{153}) > 1$ using the theorem stated above. So finally we have

Theorem. *The knots with $c(K) \leq 10$ and $u(K) = 1$ are completely determined.*

(There are 250 prime knots with $c(K) \leq 10$, and 79 have $u(K) = 1$.)

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James Heitsch, University of Illinois, Chicago, ***Index theory and non-commutative geometry***

This is joint work with M-T. Benameu. We give an alternative proof of the cohomological index theorem for families of elliptic operators defined along the leaves of a foliation of a compact manifold due to Heitsch-Lazarov. Our constructions are geometric in nature and yield highly computable invariants in Haefliger cohomology. The algebraic approach to invariants for non-commutative spaces is in some sense more fully developed than the geometric. It is our contention that the further development of the geometric approach will lend deep insight into these invariants. We expect that the application of our results will give a much fuller understanding of the relationship of the index theory of leafwise operators with the geometry and topology of the foliation. Such insight is also likely to have applications to the Novikov conjecture, one of the central problems in topology.

It is well known that the Atiyah-Singer families index theorem follows immediately from the commutative diagram

$$\begin{tikzcd}
\mathrm{K}^0_c(N) \arrow[r, "f_!"] \arrow[d, "\mathrm{ch}(\cdot)\wedge\mathrm{Td}(f)"'] & \mathrm{K}^0(M) \arrow[d, "\mathrm{ch}"] \\
\mathrm{H}^*_c(N;\mathbb{R}) \arrow[r, "f_*"] & \mathrm{H}^*(M;\mathbb{R}).
\end{tikzcd}$$

Here N and M are manifolds, with M compact. ch is the usual Chern character, and $f : N \to M$ is a K-oriented map with associated push forward map $f_!$. $\mathrm{Td}(f)$ is the Todd class of f, and f_* is the push-forward map in cohomology defined using $f_* : \mathrm{H}_*(N;\mathbb{R}) \to \mathrm{H}_*(M;\mathbb{R})$ and Poincaré Duality.

We extend this result to foliations. Let F be an oriented foliation of M and $f : N \to M/F$ a K-oriented map to the space of leaves of F. Denote by $\mathrm{H}^*_c(M/F)$ the Haefliger cohomology of F. We construct a push forward map in cohomology $f_* : \mathrm{H}^*_c(N;\mathbb{R}) \to \mathrm{H}^*_c(M/F)$. We also construct a Chern character $\mathrm{ch}^{\mathbb{R}^{2k}}_a : \mathrm{K}_0(C^\infty_c(\mathcal{G}\times\mathbb{R}^{2k})) \to \mathrm{H}^*_c(M/F)$, for k sufficiently large, where $\mathcal{G}$ is the holonomy groupoid of F. There is a Connes-Skandalis push forward map $f^{\mathbb{R}^{2k}}_! : \mathrm{K}^0_c(N) \to \mathrm{K}_0(C^\infty_c(\mathcal{G}\times\mathbb{R}^{2k}))$, for k sufficiently large. Our main result is a complete extension of the above result to foliations.

Theorem. The following diagram commutes,

$$\begin{tikzcd}
\mathrm{K}^0_c(N) \arrow[r, "f^{\mathbb{R}^{2k}}_!"] \arrow[d, "\mathrm{ch}(\cdot)\wedge\mathrm{Td}(f)"'] & \mathrm{K}_0(C^\infty_c(\mathcal{G}\times\mathbb{R}^{2k})) \arrow[d, "\mathrm{ch}^{\mathbb{R}^{2k}}_a"] \\
\mathrm{H}^*_c(N;\mathbb{R}) \arrow[r, "f_*"] & \mathrm{H}^*_c(M/F).
\end{tikzcd}$$

Denote by TF the tangent bundle of F, and by $\mathrm{Ind}_t : \mathrm{K}^0_c(TF) \to \mathrm{K}_0(C^\infty_c(\mathcal{G}\times\mathbb{R}^{2k}))$, the Connes-Skandalis topological index map. By standard methods, our theorem immediately yields a cohomological index theorem, namely

Theorem. *For any $u \in \mathrm{K}^0_c(TF)$, the algebraic Chern character of the topological index of u (in Haefliger cohomology) is given by*

$$\mathrm{ch}^{\mathbb{R}^{2k}}_a(\mathrm{Ind}_t(u)) = (-1)^p \int_F \pi_{F!}\left(\mathrm{ch}(u)\,\mathrm{Td}(TF\otimes\mathbb{C})\right) \in \mathrm{H}^*_c(M/F),$$

where $\int_F$ is integration over the leaves of F, $\pi_{F!} : H^*_c(TF;\mathbb{R}) \to \mathrm{H}^*(M;\mathbb{R})$ is integration along the fibers, $\mathrm{ch} : \mathrm{K}^0_c(TF) \to H^*_c(TF)$ is the usual Chern character, and $\mathrm{Td}(TF\otimes\mathbb{C})$ is the Todd class of the complexified foliation tangent bundle $TF\otimes\mathbb{C}$.

Theron J. Hitchman, Rice University, ***Rigidity of weakly hyperbolic standard actions of lattices in rank one Lie groups on nilmanifolds***

ABSTRACT. We give a cohomological condition sufficient for topological deformation rigidity of "standard" actions of lattices on nilmanifolds which exhibit some hyberbolic dynamics. We also discuss the problem of showing that the topological conjugacies constructed are C^∞ diffeomorphisms.

The object of study here is a lattice in a semisimple real Lie group. That is, if G is a semisimple real Lie group, a *lattice* is a subgroup Γ of G which is discrete and has co-finite volume.

We are interested in the following problem popularized by R. Zimmer [4]:

> Classify all volume preserving ergodic actions of these lattices on compact smooth manifolds by diffeomorphisms.

The known examples are very sparse, and almost all of them are obtained through algebraic constructions. As a first approach to this problem, one asks if it is possible to take one of the known algebraic actions (often called a standard action) and perturb it or deform and obtain something which is new. Most of the work so far has described conditions under which such a perturbation or deformation, if small enough, must be dynamically trivial: there is a global change of coordinates on the phase space which carries the perturbed action back to the original. This is commonly known as rigidity for the standard action.

One method is to assume that the dynamics has some type of hyperbolic behavior (like an Anosov diffeomorphism). An early result in this direction is due to S. Hurder [1].

Theorem 1 (Hurder). *Let $\Gamma < \mathbf{SL}(n;\mathbb{Z})$ be isomorphic to a higher rank lattice, and suppose that Γ contains a hyperbolic matrix. The induced standard action of Γ on $\mathbb{T}^n$ is smoothly deformation rigid.*

For lattices in groups with real rank 2 or greater, much more is now known [3].

Concurrently, further study of the internal structure of lattices has shown that lattices in the rank one group $\mathbf{Sp}(1;n)$ have many of the same properties as their higher rank cousins. Can one obtain rigidity results for standard actions of these groups, too? We have positive results in this direction. As a simple to state sample, we have

Theorem 2. *Let Γ be a lattice subgroup of $\mathbf{Sp}(1;n)$. Let $\pi : \Gamma \to \mathbf{SL}(n;\mathbb{Z})] \subset \mathbf{GL}(n;\mathbb{R})$ be an irreducible representation of Γ. Suppose that π is weakly hyperbolic, and that $H^1(\Gamma';\mathbb{R}^n_\pi) = 0$ for every finite index subgroup Γ' of Γ. Then the induced standard Type II action of Γ on the torus $M = \mathbb{R}^n/\mathbb{Z}^n$ is continuously deformation rigid.*

Many of the techniques here are inspired by Hurder's original work, and by the work of Margulis and N. Qian [3].

We also take up the question of when a such a conjugacy must be a a subgroup Γ of G which is discrete and has co-smooth diffeomorphism. The ideas here build upon work of A. Katok and J. Lewis [2], but must be modified significantly.

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John Lott, University of Michigan *Towards a synthetic notion of Ricci curvature*

For various reasons, one wishes to extend the notion of curvature to nonmanifolds. This program is successful in the case of sectional curvature, where one can make sense of a length space having Alexandrov curvature bounded below by k, by comparing geodesic triangles in the length space to geodesic triangles in a model space of constant curvature k. It is an open question if there is a good notion of Ricci curvature bounded below by k for nonmanifolds. The relevance of the question comes from Gromov's precompactness theorem for manifolds with Ricci curvature bounded below and diameter bounded above. Ideas from analysis, geometry and applied PDE seem to be converging towards an answer. We describe results of Bakry-Emery, Cheeger-Colding, Otto-Villani and the speaker.

Stephen Semmes, Rice University, *Happy fractals*

Let M be a metric space. Thus M is a nonempty set equipped with a nonnegative distance function $d(x, y)$ such that $d(x, y) = d(y, x)$ for all $x, y \in M$, $d(x, y) = 0$ if and only if $x = y$, and

$$d(x, z) \leq d(x, y) + d(y, z)$$

for all $x, y, z \in M$. We say that M is a *happy fractal* if (a) M is complete as a metric space, so that every Cauchy sequence in M converges, (b) M is doubling, in the sense that there is a $C > 0$ so that every ball in M of radius r can be covered by $\leq C$ balls of radius $\leq r/2$, and (c) there is a $C' > 0$ so that every pair of points $x, y \in M$ can be connected by a curve of length $\leq C'd(x, y)$. By a curve in M we mean a continuous mapping from an interval in the real line into M, and the length of the curve can be defined as usual as the supremum of the finite approximations to the length associated to partitions of the parameter interval.

Ordinary Euclidean spaces are happy fractals in this sense, even if they are not exactly fractal, as well as various domains in Euclidean spaces with perhaps tricky boundary behavior. The classical Sierpinksi gasket and carpet are happy fractals, and some more exotic examples are given by the Heisenberg groups and other sub-Riemannian spaces. In many cases a happy fractal M is equipped with a doubling measure μ, which is a positive Borel measure on M for which open balls have finite positive measure, and for which there is a $C'' > 0$ such that the measure of a ball with center x and radius $2r$ is less than or equal to C'' times the measure of the ball with center x and radius r. Happy fractals fit within the general framework of spaces of homogeneous type of Coifman and Weiss.

Of course lengths of curves and distribution of measure are basic themes in conformal geometry, and there are numerous connections with quasiconformal, quasiregular, and quasisymmetric mappings. This is related too to a variety of work in recent years along the lines of Sobolev spaces, rectifiable sets, and rectifiable currents in metric spaces. There have also been a number of studies concerning Laplace operators and diffusion semigroups on spaces like these. Some surveys with more information and references include "Noyaux de chaleur", a report on the trimester "Heat kernels, random walks, and analysis on manifolds and graphs" held at the Centre Émile Borel, Institut Henri Poincaré, Paris, 2002, in Gazette des Mathématiciens, "An introduction to analysis on metric spaces" and "An introduction to Heisenberg groups in analysis and geometry", in the Notices of the American Mathematical Society, and "Happy fractals and some aspects of analysis on metric spaces", in Publicacions Matemàtiques (Barcelona), all published in 2003.

Krishnan Shankar, University of Oklahoma, ***Spherical rank rigidity and Blaschke manifolds***

Most of us have an intuitive understanding of the term *curvature*. Tabletops and desktops are *flat* while basketballs and saddles are *curved*. The mathematical study of the curvature of objects is the purview of differential geometry. Geometers are able to quantify curvature precisely and it provides a numerical invariant that helps distinguish objects. My area of interest is the study of objects in dimensions higher than three that admit positive curvature. Intuitively one may think of positive curvature in the following manner: On the surface of Earth any two longitudes from the North pole appear to bend towards each other and indeed they meet at the South pole. This is true of all points on Earth if we imagine longitudes emanating from each point. Because of this, we say that the surface of Earth has positive curvature everywhere. By the same token, a saddle has negative curvature at the point where the rider sits. In higher dimensions, matters are far less visually apparent. One deals almost exclusively with equations and sophisticated geometrical techniques that describe the curvature of *manifolds*, a term that refers to objects that, roughly speaking, have no sharp edges. Manifolds of bounded size are called *compact* manifolds. One of the great mysteries in the study of positive curvature is the dearth of examples of manifolds that have positive curvature at every point. The techniques at hand are few and the number of known examples remains relatively small. In fact, even compact manifolds with non-negative curvature (a class which contains all positively curved manifolds) are not very well understood.

In recent joint work with Ralf Spatzier (of the University of Michigan) and Burkhard Wilking (of the University of Munster, Germany), we introduced the notion of spherical rank and proved a rigidity theorem for the same. A rigidity theorem is one which pins down the structure of a manifold given certain geometric conditions. For instance, the Sphere theorem says that any manifold with curvature at least 1 and diameter bigger than $\pi/2$ must be (homeomorphic to) a sphere. This is a very strong structural result about any manifold satisfying certain geometric condition. One of the most fundamental themes in geometry in particular is the proving of rigidity theorems. Our result may be cast in the general framework of rigidity results like the Sphere theorem (for positive curvature) or the Mostow Rigidity theorem (for non-positive curvature). Indeed, the motivation for the problem came from analogous rigidity theorems in non-positive curvature (for Euclidean rank or hyperbolic rank).

Given a manifold we can measure distances between a pair of points by measuring the length of the shortest curve between them. For instance, on the sphere of radius 1 the distance between the North pole and the South pole is π, which is the length of any longitude. Similarly, the distance from the North pole to a point on the Equator is $\pi/2$ and this distance is realized by a (unique) piece of a longitude. Such a curve which realizes the shortest distance between two points is called a *geodesic*. In a flat space geodesics are simply straight lines as one might expect, but on more general manifolds geodesics can be fairly complicated. On a sphere, notice that if two points are at distance less than π, then there is a unique geodesic between them. On the other hand, if they are at distance π (like the two poles) then there are infinitely many geodesics connecting them, namely any longitude has length π. The smallest number for which this happens is called the *injectivity radius* of the manifold (strictly speaking this is not correct but this definition will do for now). We just argued that the injectivity radius of the sphere of radius 1 is π. For a flat plane, there is always a unique line segment connecting any two points, so the injectivity radius of a flat plane (or flat three dimensional space) is ∞.

The *diameter* of a manifold is defined in a fairly intuitive fashion; it is simply the maximal separation between a pair of points on the manifold. A sphere of radius 1 has diameter π; this fact is not completely obvious but not too hard to prove. A flat plane has infinite diameter since there is no bound on the distance between any two points. In fact a manifold is compact precisely when its diameter is finite.

The concepts of injectivity radius and diameter were introduced early in the twentieth century to understand better the structure of manifolds. A *Blaschke manifold* is one in which the injectivity radius equals the diameter. For instance, the sphere of radius 1 is indeed a Blaschke manifold. The definition of a Blaschke manifold may seem unmotivated but there were good reasons for Blaschke to define them so; he was incidentally one of the greatest geometers of the twentieth century. He originally called them *wiedersehn manifolds*; this is German for, "see you again" which reflected the geometric fact that the geometry near any point looked like the geometry at some point a fixed distance away i.e., in some sense you "see" the point again. Blaschke manifolds are believed to be very rigid which is expressed as the following:

Blaschke Conjecture. *A simply connected Blaschke manifold is isometric to a compact, rank one symmetric space.*

The conjecture remains unresolved to this day, however it is known to be true in certain special cases. The compact, rank one symmetric spaces fall into one of four special types of manifolds (one of the types is the round sphere as one might expect) and are completely understood, so the Blaschke conjecture is a rigidity result of the strongest possible type. In our paper we made an assumption on the geometry of the manifold; we assumed that the maximum of all curvatures is 1 and we assumed that along every geodesic we hit a conjugate point at distance π; i.e., along every geodesic there is at least one direction orthogonal to the geodesic along which it looks infinitesimally like the sphere of radius 1. In this case we say that the manifold has spherical rank at least 1. Under these conditions we showed that the manifold is a Blaschke manifold and moreover, it is isometric to a compact, rank one symmetric space. Namely, we prove a particular case of the Blaschke conjecture.

While the result is optimal in a strong sense, this leads to several interesting questions about weakened notions of spherical rank. This is now an active part of our research program.

Mikhail Shubin, Northeastern University, *Spectral properties of Schrödinger operators*

Friedrichs proved in 1934 that the spectrum of the Schrödinger operator $H_V = -\Delta + V$ in $L^2(\mathbf{R}^n)$ with a locally integrable potential V is discrete provided $V(x) \to +\infty$ as $|x| \to \infty$. On the other hand, if we assume that $V \geq 0$ (or V is semi-bounded below), then it is possible to give a necessary and sufficient condition for the discreteness of spectrum using Wiener capacity (A. M. Molchanov, 1953):

$$\inf_F \int_{Q_d \setminus F} V(x)\, dx \to +\infty \quad \text{as} \quad Q_d \to \infty, \tag{1}$$

where Q_d is a closed cube with the edge length d and with the edges parallel to coordinate axes, $Q_d \to \infty$ means that the cube Q_d goes to infinity (with fixed d), and the infimum is taken over all compact subsets F of Q_d which are called *n*egligible. The negligibility of F in the sense of Molchanov means that $\mathrm{cap}(F) \leq \gamma\ \mathrm{cap}(Q_d)$, where cap is the Wiener capacity and $\gamma > 0$ is a specific small constant $\gamma = c_n$. In 1953 I. M. Gelfand raised the question about the range of all possible constants γ This question was recently answered in a joint paper by V. Maz'ya and M. Shubin (2003): it was proven that we can take any $\gamma \in (0, 1)$.

This result has the following striking corollary: if we take two different constants $\gamma \in (0, 1)$, e.g. $\gamma = 0.01$ and $\gamma = 0.99$, then the corresponding conditions (1) will be equivalent. So if the condition (1) is satisfied for the class of negligible sets F which occupy at most 0.01 of capacity of the cube Q_d, then it also holds with F's occupying 0.99 of the capacity of Q_d.

The cubes here can be replaced by more general bodies (e.g. images of a fixed convex body under homotheties and translations). Moreover, γ can be taken dependent on d and the class of all possible functions $\gamma(d)$ is completely described.

Similar results are obtained for the strict positivity problem, where new necessary and sufficient conditions of zero-in-the-spectrum property of H_V with $V \geq 0$ are given.

The results above also extend to the Schrödinger operators in arbitrary open sets $\Omega \subset \mathbf{R}^n$, with the Dirichlet boundary conditions on the boundary of Ω.

The same methods allow to obtain two-sided estimates for the bottom of the spectrum and essential spectrum of H_V. In particular, they apply to the Laplacian in arbitrary (bounded or unbounded) domains $\Omega \subset \mathbf{R}^n$. These estimates are given in terms of a geometric characteristic of Ω called interior capacitary diameter, which in turn depends on the choice of the class of the negligible sets above, though the corresponding estimates with different constants $\gamma \in (0, 1)$ prove to be equivalent.

Magdalena Toda, Texas Tech University, *Lorentz conformal structures and the wave equation*

This report is based on the presenter's most recent research on Lorentz surfaces, and some joint work with J. Inoguchi. The main result consists of a general representation of timelike minimal surfaces. As a topic, this is perfectly integrated in the author's general study area, namely algebro-geometric representations of Riemannian and Lorentzian 2-dimensional manifolds, as conformal structures.

The classical wave equation:

$$-u_{tt} + u_{xx} = 0$$

over a plane $\mathbb{R}^2(t, x)$ can be understood as the *harmonicity equation*with respect to the Lorentz metric $-dt^2 + dx^2$. It is easy to see that Lorentz harmonicity is invariant under conformal transformations.

Thus, in order to study geometric solutions to second order PDEs of hyperbolic type, it is convenient to introduce the notion of *Lorentz conformal structure*. Oriented 2-manifolds together with a Lorentz conformal structure are called *Lorentz surfaces*.

R. Kulkarni initiated the global study of Lorentz conformal structure and Lorentz surfaces. Many important contributions to the field were brought by T. Weinstein and her collaborators.

Among Lorentz surfaces, timelike minimal surfaces play an important role. The fundamentals of classical string theory can be summarized as follows: A *closed string* is an object γ in the physical space, that is homeomorphic to S^1. Intuitively speaking, a string evolves in time while sweeping a surface Σ, called *world sheet*, in spacetime. For physical reasons, Σ is supposed to be a timelike surface. The dynamical equations for a string are defined by a variational principle: The first area variation of Σ must vanish subject to the condition that the initial and final configuration of the string are kept fixed. Hence, Σ will be a timelike minimal surface having two spacelike boundary components.

It is easy to see that timelike minimal surfaces in a spacetime can be realized via conformal harmonic maps of Lorentz surfaces. Conformal harmonic maps of a Lorentz surface into a semi-Riemannian symmetric space G/K are called *nonlinear sigma models* (with symmetry group G) in particle physics, and in some other fields, *wave maps*. Nonlinear sigma models may be regarded as toy models of gauge theory.

Studying such surfaces has a genuine impact on applied sciences, e.g., this timelike minimal surface theory applies to fluid dynamics.

The present work started with a systematic study of Lorentz conformal structure from structural viewpoints. It turned out that split complex numbers and split quaternions are extremely useful tools in the conformal realization of Lorentz surfaces. For both physical and geometric reasons (comparison with their Euclidean counterparts), the presenter focuses on minimal ones.

The central result is a loop group representation and a construction method for all the timelike minimal surfaces in the Lorentz 3-space, based on this representation.