Cruel Summer: A Type D Asymmetric Simple Exclusion Process Generated by an Explicit Central Element of $\mathcal{U}_q(\mathfrak{so}_{10})$

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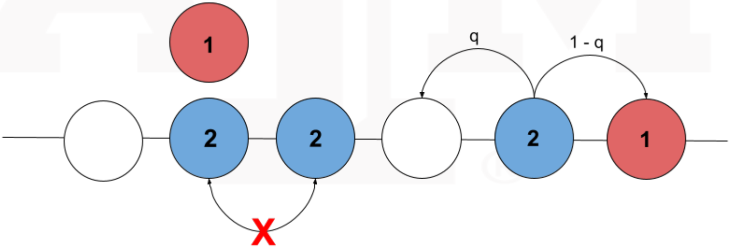
Overview

Begin Again

Type D ASEP

Type D ASEP (asymmetric simple exclusion process):

1. There are two classes of particles (class 1 and class 2).
2. At most two particles can occupy a site, and they must be of different classes
3. Particles drift left at a different rate than it drifts to the right
4. There are three parameters (q, n, δ)



Transition Matrices

Definition

The **transition matrices** of a Markov process X_t are matrices P(t) with rows and columns indexed by the state space $\mathfrak{X}$ with **transition probabilities**:

$$p_{xy}(t,s) = \mathbb{P}(X_{s+t} = y | X_s = x)$$

If $p_{xy}(t,s)$ does not depend on s, we say X_t is **time-homogeneous** and write $p_{xy}(t)$

Example

Transition matrix of simple random walk:

$$P = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}$$

Lie algebras

Definition

A **Lie algebra** is a complex vector space $\mathfrak{g}$ along with a **bracket** operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that the following hold:

- $[X, Y] = -[Y, X]$ (*skew-symmetry*)
- $[X, aY + bZ] = a[X, Y] + b[X, Z]$ (*bilinearity*)
- $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$ (*Jacobi's identity*)

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Example

The *special orthogonal Lie algebra* is the space of $2n \times 2n$ block complex matrices

$$\mathfrak{so}_{2n} = \left\{ \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} : A, B, C \in M_{n\times n}(\mathbb{C}), B = -B^T, C = -C^T \right\}$$

with bracket $[X, Y] = XY - YX$.

Weights and roots

Main idea: **Weights** are generalized versions of eigenvalues. A **weight space** is like an eigenspace.

A **representation** is a way of assigning how each element in $\mathfrak{so}_{2n}$ acts on vectors. So weights and weight spaces depend on which representation we are talking about.

Important representations

- Fundamental representation
  - Weights are called **fundamental weights**, denoted μ or λ
  - Elements of the weight space are denoted v_μ or v_λ
- Adjoint representation
  - Weights are called **roots** and denoted $\pm\alpha$
  - Half of these roots are designated to be "positive"

Weights and roots

Let L_i be the function sending a matrix to its i-th diagonal entry. Then the fundamental weights and roots for $\mathfrak{so}_{2n}$ are

$$\mu = \pm L_i \quad \alpha = \pm L_i \pm L_j$$

for $1 \leq i < j \leq n$.

Universal enveloping algebras

In a Lie algebra $\mathfrak{g}$, you cannot multiply elements; you can only bracket them.

We want to allow multiplication as well as the Lie bracket, so we embed into a new space called the **universal enveloping algebra** $\mathcal{U}(\mathfrak{g})$.

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Example

$\mathcal{U}(\mathfrak{so}_{2n})$ is generated by matrices E_i, F_i, H_i $(1 \leq i \leq n)$, with certain relationships between them. For example, for $n = 2$:

$$E_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \; F_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \; H_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Quantum groups

The objects we are actually studying are "q-deformed" versions of these universal enveloping algebras, called the **quantum group** $\mathcal{U}_q(\mathfrak{g})$. Here, $q > 0$ is a parameter that will end up having significance on the probability side.

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Example

$\mathcal{U}_q(\mathfrak{so}_{2n})$ is generated by E_i, F_i, q^{H_i} $(1 \leq i \leq n)$, with q-deformed relationships between them. For example:

$$[E_i, F_i] = \frac{q^{H_i} - q^{-H_i}}{q - q^{-1}}$$

Quantum groups

The quantum group $\mathcal{U}_q(\mathfrak{so}_{2n})$ also has a coproduct $\Delta : \mathcal{U}_q(\mathfrak{so}_{2n}) \to \mathcal{U}_q(\mathfrak{so}_{2n}) \otimes \mathcal{U}_q(\mathfrak{so}_{2n})$ which will allow us translate between algebra and probability.

Example

The coproducts of the generators are

$$\Delta(E_i) = E_i \otimes 1 + q^{H_i} \otimes E_i, \quad \Delta(F_i) = 1 \otimes F_i + F_i \otimes q^{-H_i}, \quad \Delta(q^{H_i}) = q^{H_i} \otimes q^{H_i}.$$

Problem: Computing a central element for $\mathcal{U}_q(\mathfrak{so}_{10})$

During the 2020 REU, students computed central elements for $\mathcal{U}_q(\mathfrak{so}_6)$ and $\mathcal{U}_q(\mathfrak{so}_8)$ and constructed the associated Markov processes.

This Summer

1. Used Python to compute a central element for $\mathcal{U}_q(\mathfrak{so}_{10})$
2. Created the associated generator matrix for the Markov process
3. Verified that our matrix generated a Type D ASEP as conjectured
4. Extended a Duality result about the Type D ASEP from [BBKUZ22]

The Last Great American Dynasty: A Central Element of $\mathcal{U}_q(\mathfrak{so}_{10})$

Motivating formula

We use the following lemma from [Kua16], which was based on [Jan95].

Lemma

For each fundamental weight μ, let v_μ be a vector in its weight space. Given weights $\mu > \lambda$, suppose $e_{\mu\lambda}$ sends v_λ to v_μ and $f_{\lambda\mu}$ sends v_μ to v_λ. If $e^*_{\mu\lambda}, f^*_{\lambda\mu}$ are their q-pairing dual elements, and ρ is half the sum of the positive roots of $\mathfrak{so}_{2n}$, then

$$\sum_{\mu} q^{(-2\rho,\mu)} q^{H_{-2\mu}} + \sum_{\mu>\lambda} q^{(\mu-\lambda,\mu)} q^{(-2\rho,\mu)} e^*_{\mu\lambda} q^{H_{-\mu-\lambda}} f^*_{\lambda\mu}$$

is a central element of $\mathcal{U}_q(\mathfrak{so}_{2n})$.

Weights and roots

Lemma

For each ***fundamental weight*** μ, ***let*** v_μ ***be a vector in its weight space****. Given weights* $\mu > \lambda$*, suppose* $e_{\mu\lambda}$ *sends* v_λ *to* v_μ *and* $f_{\lambda\mu}$ *sends* v_μ *to* v_λ*. If* $e^*_{\mu\lambda}, f^*_{\lambda\mu}$ *are their q-pairing dual elements, and* ρ ***is half the sum of the positive roots of*** $\mathfrak{so}_{2n}$*, then*

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Easy terms

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Easy terms

$$\sum_{\mu} q^{(-2\rho,\mu)} q^{H_{-2\mu}} + \sum_{\mu>\lambda} q^{(\mu-\lambda,\mu)} q^{(-2\rho,\mu)} e^*_{\mu\lambda} q^{H_{-\mu-\lambda}} f^*_{\lambda\mu}$$

- $(-2\rho,\mu)$ and $(\mu-\lambda,\mu)$ are just the usual dot product, so these terms are just powers of q, which are easy to compute
- $q^{H_{-2\mu}}$ and $q^{H_{-\mu-\lambda}}$ are just products of $q^{\pm H_i}$, which are also easy to compute

Ordering the weights

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For each fundamental weight μ, let v_μ be a vector in its weight space. Given weights $\mu > \lambda$, suppose $e_{\mu\lambda}$ sends v_λ to v_μ and $f_{\lambda\mu}$ sends v_μ to v_λ. If $e^*_{\mu\lambda}, f^*_{\lambda\mu}$ are their q-pairing dual elements, and ρ is half the sum of the positive roots of $\mathfrak{so}_{2n}$, then

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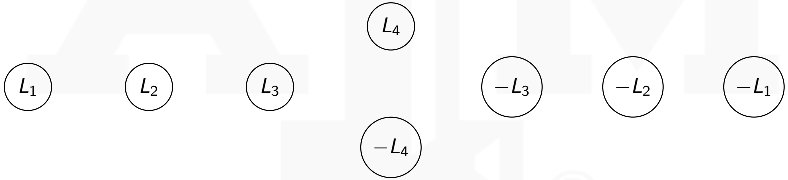
Ordering the weights

We order the weights by

$$L_1 > \cdots > L_{n-1} > L_n = -L_n > -L_{n-1} > \cdots > -L_1.$$

We can visualize this as (for $n = 4$):



$e_{\mu\lambda}$ and $f_{\lambda\mu}$

Lemma

For each fundamental weight μ, let v_μ be a vector in its weight space. Given weights $\mu > \lambda$, ***suppose $e_{\mu\lambda}$ sends v_λ to v_μ and $f_{\lambda\mu}$ sends v_μ to v_λ****. If $e^*_{\mu\lambda}, f^*_{\lambda\mu}$ are their q-pairing dual elements, and ρ is half the sum of the positive roots of $\mathfrak{so}_{2n}$, then*

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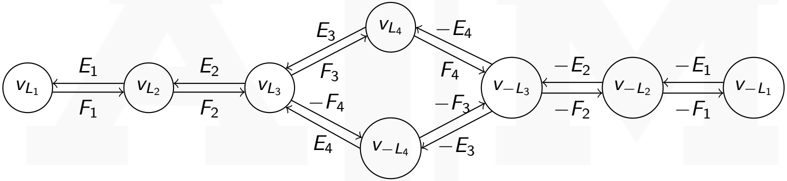
$e_{\mu\lambda}$ and $f_{\lambda\mu}$

The E_i and F_i act on weight spaces in the following way:



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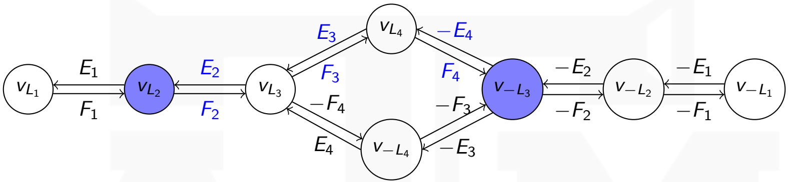

Example

Consider $\mu = L_2, \lambda = -L_3$.

- $e_{L_2,-L_3}$ sends v_{-L_3} to v_{L_2}, so

$$e_{L_2,-L_3} = (E_2)(E_3)(-E_4)$$

- f_{-L_3,L_2} sends v_{L_2} to v_{-L_3}, so

$$f_{-L_3,L_2} = (F_4)(F_3)(F_2)$$

q-pairing

Lemma

*For each fundamental weight μ, let v_μ be a vector in its weight space. Given weights $\mu > \lambda$, suppose $e_{\mu\lambda}$ sends v_λ to v_μ and $f_{\lambda\mu}$ sends v_μ to v_λ. If $e^*_{\mu\lambda}, f^*_{\lambda\mu}$ are their **q-pairing** dual elements, and ρ is half the sum of the positive roots of $\mathfrak{so}_{2n}$, then*

$$\sum_{\mu} q^{(-2\rho,\mu)} q^{H_{-2\mu}} + \sum_{\mu>\lambda} q^{(\mu-\lambda,\mu)} q^{(-2\rho,\mu)} e^*_{\mu\lambda} q^{H_{-\mu-\lambda}} f^*_{\lambda\mu}$$

is a central element of $\mathcal{U}_q(\mathfrak{so}_{2n})$.

q-pairing

We have a "q-deformed pairing" $\langle \cdot, \cdot \rangle$ which acts as follows:

$$\langle \text{product of } F_i\text{'s and } q^{\pm H_i}\text{'s}, \text{product of } E_i\text{'s and } q^{\pm H_i}\text{'s} \rangle = \text{rational function in } q$$

These can be computed inductively on the number of terms.

Dual elements

Lemma

*For each fundamental weight μ, let v_μ be a vector in its weight space. Given weights $\mu > \lambda$, suppose $e_{\mu\lambda}$ sends v_λ to v_μ and $f_{\lambda\mu}$ sends v_μ to v_λ. If $e^*_{\mu\lambda}, f^*_{\lambda\mu}$ **are their q-pairing dual elements**, and ρ is half the sum of the positive roots of $\mathfrak{so}_{2n}$, then*

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Dual elements

Say we want to find the dual element to $f_{-L_3,L_2} = F_4 F_3 F_2 = F_{432}$. (Note: we often abbreviate $F_a F_b F_c$ as F_{abc}.)

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Step 1: Find all permutations of the index set $\{4,3,2\}$:

$$F_{432}, F_{423}, F_{234}, F_{243}, F_{324}, F_{342}$$

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Step 1: Find all permutations of the index set $\{4, 3, 2\}$:

$$F_{432}, F_{423}, F_{234}, F_{243}, F_{324}, F_{342}$$

Step 2: Many of these could be linearly dependent due to the relationships between the F_i. Find a basis that is linearly independent:

$$F_{432}, F_{423}, F_{324}, F_{243}$$

Call these f_1, f_2, f_3, f_4.

Dual elements

Step 3: Make a corresponding basis of products of E_i's:

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Dual elements

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Call these e_1, e_2, e_3, e_4.

Step 4: Form a matrix M of q-pairings by $M_{ij} = \langle f_i, e_j \rangle$:

$$M = (q - q^{-1})^{-3} \begin{pmatrix} 1 & 1/q & 1/q & q^{-2} \\ 1/q & 1 & q^{-2} & 1/q \\ 1/q & q^{-2} & 1 & 1/q \\ q^{-2} & 1/q & 1/q & 1 \end{pmatrix}$$

Dual elements

Step 5: Invert the matrix:

$$M^{-1} = (q - q^{-1}) \begin{pmatrix} q^2 & -q & -q & 1 \\ -q & q^2 & 1 & -q \\ -q & 1 & q^2 & -q \\ 1 & -q & -q & q^2 \end{pmatrix}$$

Dual elements

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Step 6: Read off the row corresponding to the element we're interested in. We want f_1^*, so we use the first row:

$$f_1^* = (q - q^{-1})(q^2 e_1 - q e_2 - q e_3 + e_4)$$

Summary

Lemma

For each fundamental weight μ, let v_μ be a vector in its weight space. Given weights $\mu > \lambda$, suppose $e_{\mu\lambda}$ sends v_λ to v_μ and $f_{\lambda\mu}$ sends v_μ to v_λ. If $e^*_{\mu\lambda}, f^*_{\lambda\mu}$ are their q-pairing dual elements, and ρ is half the sum of the positive roots of $\mathfrak{so}_{2n}$, then

$$\sum_{\mu} q^{(-2\rho,\mu)} q^{H_{-2\mu}} + \sum_{\mu>\lambda} q^{(\mu-\lambda,\mu)} q^{(-2\rho,\mu)} e^*_{\mu\lambda} q^{H_{-\mu-\lambda}} f^*_{\lambda\mu}$$

is a central element of $\mathcal{U}_q(\mathfrak{so}_{2n})$.

The Central Element of $\mathcal{U}_q(\mathfrak{so}_{10})$

Computing the Central Element

- We began by coding the described method (a large amount of the code comes from [KLLPZ20]).
- It is difficult to symbolically find the determinant or inverse of large matrices

$\mathcal{U}_q(\mathfrak{so}_{10})$ Numerically

The Process:

1. Plug in a value for q
2. Determine the bases for the dual elements
3. Determine the dual elements in terms of q

Example (q = 10)

$10000F_{12354} - 1000F_{12435} - 1000F_{12534} + 100F_{12543} - 1000F_{13254} + 100F_{14325} + 100F_{15324} - 10F_{15432} - 1000F_{21354} + 100F_{21435} + 100F_{21534} - 10F_{21543} + 100F_{32154} - 10F_{43215} - 10F_{53214} + F_{54321}$

Central Element

First, for ease of reading we set some notational shortcuts following the convention in [KLLPZ20]. Set $r = q - \frac{1}{q}$, and let $K_i = q^{H_i}$

Theorem

The following element is in the center of $\mathcal{U}_q(\mathfrak{so}_{10})$:

$$q^8 K_{11223345} + q^6 K_{223345} + q^4 K_{3345} + q^2 K_{45} + K_4 K_5^{-1} + K_4^{-1} K_5 + \frac{1}{q^2} K_{45}^{-1} + \frac{1}{q^4} K_{3345}^{-1} + \frac{1}{q^6} K_{223345}^{-1} + \frac{1}{q^8} K_{11223345}^{-1}$$
$$+ \frac{r^2}{q} F_4 K_5^{-1} E_4 + \frac{r^2}{q^3}(qF_{34} - F_{43}) K_{35}^{-1}(qE_{43} - E_{34}) + \frac{r^2}{q^3} F_3 K_{345}^{-1} E_3$$
$$+ \frac{r^2}{q^5}(q^2 F_{234} - qF_{243} - qF_{324} + F_{432}) K_{235}^{-1}(q^2 E_{432} - qE_{342} - qE_{423} + E_{234})$$
$$+ \frac{r^2}{q^5}(qF_{23} - F_{32}) K_{2345}^{-1}(qE_{32} - E_{23}) + \frac{r^2}{q^5} F_2 K_{23345}^{-1} E_2 + \frac{r^2}{q^7}(\boxed{A_1}) K_{1235}^{-1}(\boxed{A_2})$$
$$+ \frac{r^2}{q^7}(q^2 F_{123} - qF_{132} - qF_{213} + F_{321}) K_{12345}^{-1}(q^2 E_{321} - qE_{231} - qE_{312} + E_{123})$$
$$+ \frac{r^2}{q^7}(qF_{12} - F_{21}) K_{123345}^{-1}(qE_{21} - E_{12}) + \frac{r^2}{q^7} F_1 K_{1223345}^{-1} E_1 - qr^2(\boxed{A_3}) K_{1234}(\boxed{A_4})$$

Central Element

Theorem

$$
\begin{aligned}
&- qr^2(q^2F_{532} - qF_{352} - qF_{523} + F_{235})K_{234}(q^2E_{235} - qE_{253} - qE_{325} + E_{532})\\
&- qr^2(q^2F_{532} - qF_{352} - qF_{523} + F_{235})K_{234}(q^2E_{235} - qE_{253} - qE_{325} + E_{532})\\
&- qr^2(qF_{53} - F_{35})K_{34}(qE_{35} - E_{53}) - qr^2F_5K_4E_5 - r^4F_{54}E_{54} - \frac{r^2}{q}(\boxed{A_5})K_{123}(\boxed{A_6}) - \frac{r^2}{q}(\boxed{A_7})K_{23}(\boxed{A_8})\\
&- \frac{r^2}{q}(q^2F_{453} - qF_{435} - qF_{534} + F_{345})K_3(q^2E_{354} - qE_{435} - qE_{534} + E_{543}) - \frac{r^2}{q}F_5K_4^{-1}E_5\\
&- \frac{r^4}{q^2}(-qF_{3435} - qF_{5343} + (q^2+1)F_{3543})(-qE_{3435} - qE_{5343} + (q^2+1)E_{3543}) - \frac{r^2}{q^3}(\boxed{A_9})K_2(\boxed{A_{10}})\\
&- \frac{r^2}{q^3}(q^2F_{354} - qF_{435} - qF_{534} + F_{543})K_3^{-1}(q^2E_{453} - qE_{435} - qE_{534} + E_{345}) - \frac{r^2}{q^3}(qF_{35} - F_{53})K_{34}^{-1}(qE_{53} - E_{35})\\
&- \frac{r^2}{q^3}(\boxed{A_{11}})K_{12}(\boxed{A_{12}}) - \frac{r^4}{q^4}(\boxed{A_{13}})(\boxed{A_{14}}) - \frac{r^2}{q^5}(\boxed{A_{15}})K_2^{-1}(\boxed{A_{16}}) - \frac{r^2}{q^5}(\boxed{A_{17}})K_{23}^{-1}(\boxed{A_{18}})
\end{aligned}
$$

Central Element

Theorem

$$- \frac{r^2}{q^5}(q^2 F_{235} - qF_{253} - qF_{325} + F_{532})K_{234}^{-1}(q^2 E_{532} - qE_{352} - qE_{523} + E_{235}) - \frac{r^2}{q^5}(\boxed{A_{19}})K_1(\boxed{A_{20}})$$

$$- \frac{r^4}{q^6}(\boxed{A_{21}})(\boxed{A_{22}}) - \frac{r^2}{q^7}(\boxed{A_{23}})K_{123}^{-1}(\boxed{A_{24}}) - \frac{r^2}{q^7}(\boxed{A_{25}})K_{1234}^{-1}(\boxed{A_{26}}) - \frac{r^2}{q^7}(\boxed{A_{27}})K_{12}^{-1}(\boxed{A_{28}})$$

$$- \frac{r^2}{q^7}(\boxed{A_{29}})K_1^{-1}(\boxed{A_{30}})q^7 r^2 F_1 K_{1223345} E_1 + q^5 r^2 (qF_{21} - F_{12})K_{123345}(qE_{12} - E_{21}) + q^5 r^2 F_2 K_{23345} E_2$$

$$+ q^3 r^2 (q^2 F_{321} - qF_{231} - qF_{312} + F_{123})K_{12345}(q^2 E_{123} - qE_{132} - qE_{213} + E_{321})$$

$$+ q^3 r^2 (qF_{32} - F_{23})K_{2345}(qE_{23} - E_{32}) + q^3 r^2 F_3 K_{345} E_3 + qr^2(\boxed{A_{31}})K_{1235}(\boxed{A_{32}})$$

$$+ qr^2 (q^2 F_{432} - qF_{342} - qF_{423} + F_{234})K_{235}(q^2 E_{234} - qE_{243} - qE_{324} + E_{432})$$

$$+ qr^2 (qF_{43} - F_{34})K_{35}(qE_{34} - E_{43}) + qr^2 F_4 K_5 E_4 ,$$

Central Element

Theorem

where

$$
\begin{aligned}
A_1 &= q^3 F_{1234} - q^2 F_{1243} - q^2 F_{1324} - q^2 F_{2134} + qF_{1432} + qF_{2143} + qF_{3214} - F_{4321}\\
A_2 &= q^3 E_{4321} - q^2 E_{3421} - q^2 E_{4231} - q^2 E_{4312} + qE_{2341} + qE_{3412} + qE_{4123} - E_{1234}\\
A_3 &= q^3 F_{5321} - q^2 F_{3521} - q^2 F_{5231} - q^2 F_{5312} + qF_{2351} + qF_{3512} + qF_{5123} - F_{1235}\\
A_4 &= q^3 E_{1235} - q^2 E_{1253} - q^2 E_{1325} - q^2 E_{2135} + qE_{1532} + qE_{2153} + qE_{3215} - E_{5321}\\
A_5 &= q^4 F_{45321} - q^3 F_{43521} - q^3 F_{45231} - q^3 F_{45312} - q^3 F_{53421} + q^2 F_{34521} + q^2 F_{42351} + q^2 F_{43512}\\
&\quad + q^2 F_{45123} + q^2 F_{52341} + q^2 F_{53412} - qF_{23451} - qF_{34512} - qF_{41235} - qF_{51234} + F_{12345}\\
A_6 &= q^4 E_{12354} - q^3 E_{12435} - q^3 E_{12534} - q^3 E_{13254} - q^3 E_{21354} + q^2 E_{12543} + q^2 E_{14325} + q^2 E_{15324}\\
&\quad + q^2 E_{21435} + q^2 E_{21534} + q^2 E_{32154} - qE_{15432} - qE_{21543} - qE_{43215} - qE_{53214} + E_{54321}\\
A_7 &= q^3 F_{4532} - q^2 F_{4352} - q^2 F_{4523} - q^2 F_{5342} + qF_{3452} + qF_{4235} + qF_{5234} - F_{2345}\\
A_8 &= q^3 E_{2354} - q^2 E_{2435} - q^2 E_{2534} - q^2 E_{3254} + qE_{2543} + qE_{4325} + qE_{5324} - E_{5432},
\end{aligned}
$$

Central Element

Theorem

and,

$$
\begin{aligned}
A_9 &= q^4 F_{34532} - (q^3 - q)F_{35342} - (q^3 - q)F_{43532} + q^2 F_{34235} + q^2 F_{35234} + q^2 F_{43523} \\
&\quad + q^2 F_{53423} - qF_{32345} - qF_{43235} - qF_{45323} - qF_{53234} + F_{23453} + (-q^3 - q)F_{34523} \\
A_{10} &= q^4 E_{23543} - q^3 E_{23435} - q^3 E_{25343} + q^2 E_{32435} + q^2 E_{32534} + q^2 E_{43253} + q^2 E_{53243} \\
&\quad - qE_{34325} - qE_{53432} + E_{35432} + (-q^3 - q)E_{32543} \\
A_{11} &= q^5 F_{354321} - q^4 F_{354312} - (q^4 - q^2)F_{343521} - (q^4 - q^2)F_{534321} + q^3 F_{342351} + q^3 F_{352341} \\
&\quad + q^3 F_{435231} + q^3 F_{534231} - q^2 F_{323541} - q^2 F_{341235} - q^2 F_{351234} - q^2 F_{432351} - q^2 F_{435123} \\
&\quad - q^2 F_{532341} - q^2 F_{534123} - q^2 F_{543231} + qF_{235431} + qF_{312354} + qF_{431235} + qF_{531234} \\
&\quad + qF_{543123} + (q^3 - q)F_{343512} + (q^3 - q)F_{534312} + (q^3 + q)F_{354123} + (-q^4 - q^2)F_{354231} - F_{123543} \\
A_{12} &= q^5 E_{123543} - q^4 E_{123435} - q^4 E_{125343} - q^4 E_{213543} + q^3 E_{132435} + q^3 E_{132534} + q^3 E_{143253} \\
&\quad + q^3 E_{153243} + q^3 E_{213435} + q^3 E_{215343} - q^2 E_{134325} - q^2 E_{153432} - q^2 E_{321435} - q^2 E_{321534} - q^2 E_{432153} \\
&\quad - q^2 E_{532143} + qE_{135432} + qE_{343215} + qE_{534321} + (q^3 + q)E_{321543} + (-q^4 - q^2)E_{132543} - E_{354321},
\end{aligned}
$$

Central Element

Theorem

and,

$$
\begin{aligned}
A_{13} &= q^2 F_{235234} + q^2 F_{243523} - q^2 F_{223543} + q^2 F_{324352} + q^2 F_{532432} + (-q^3 - q) F_{234352} \\
&\quad + (-q^3 - q) F_{253432} + (-q^3 - q) F_{325432} + (q^4 + q^2 + 1) F_{235432} \\
A_{14} &= q^2 E_{235234} + q^2 E_{243523} - q^2 E_{223543} + q^2 E_{324352} + q^2 E_{532432} + (-q^3 - q) E_{234352} \\
&\quad + (-q^3 - q) E_{253432} + (-q^3 - q) E_{325432} + (q^4 + q^2 + 1) E_{235432}, \\
A_{15} &= q^4 F_{23543} - q^3 F_{23435} - q^3 F_{25343} + q^2 F_{32435} + q^2 F_{32534} + q^2 F_{43253} + q^2 F_{53243} \\
&\quad - q F_{34325} - q F_{53432} + F_{35432} + (-q^3 - q) F_{32543} \\
A_{16} &= q^4 E_{34532} - (q^3 - q) E_{35342} - (q^3 - q) E_{43532} + q^2 E_{34235} + q^2 E_{35234} + q^2 E_{43523} + q^2 E_{53423} \\
&\quad - q E_{32345} - q E_{43235} - q E_{45323} - q E_{53234} + E_{23453} + (-q^3 - q) E_{34523} \\
A_{17} &= q^3 F_{2354} - q^2 F_{2435} - q^2 F_{2534} - q^2 F_{3254} + q F_{2543} + q F_{4325} + q F_{5324} - F_{5432} \\
A_{18} &= q^3 E_{4532} - q^2 E_{4352} - q^2 E_{4523} - q^2 E_{5342} + q E_{3452} + q E_{4235} + q E_{5234} - E_{2345},
\end{aligned}
$$

Central Element

Theorem

and,

$$\begin{aligned}A_{19} &= q^6F_{2354321} + q^4F_{2343512} + q^4F_{2534312} + (q^4 - q^2)F_{2352341} + (q^4 - q^2)F_{2435231}\\ &- (q^4 - q^2)F_{2235431} + (q^4 - q^2)F_{3243521} + (q^4 - q^2)F_{5324321} - q^3F_{2341235} - q^3F_{2351234}\\ &- q^3F_{2435123} - q^3F_{2534123} - q^3F_{3243512} - q^3F_{3253412} - q^3F_{4325312} - q^3F_{5324312} + q^2F_{2312354}\\ &+ q^2F_{2431235} + q^2F_{2531234} + q^2F_{2543123} + q^2F_{3241235} + q^2F_{3251234} + q^2F_{3432512} + q^2F_{4325123}\\ &+ q^2F_{5324123} + q^2F_{5343212} - qF_{2123543} - qF_{3212354} - qF_{3543212} - qF_{4321235}\\ &- qF_{5321234} - qF_{5432123} + (-q^3 - q)F_{3254123} + (q^4 + q^2)F_{2354123} + (q^4 + q^2)F_{3254312}\\ &+ (-q^5 - q)F_{2354312} + (-q^5 + q)F_{2343521} + (-q^5 + q)F_{2534321} + (-q^5 + q)F_{3254321} + F_{1235432}\\ A_{20} &= q^6E_{1235432} - q^5E_{1234352} - q^5E_{1253432} - q^5E_{1325432} + q^4E_{1235234} + q^4E_{1243523}\\ &+ q^4E_{1324352} + q^4E_{1532432} - (q^4 - q^2)E_{1223543} - q^3E_{1235423} - q^3E_{2134235} - q^3E_{2135234}\\ &- q^3E_{2143523} - q^3E_{2153423} - q^3E_{2354213} - q^3E_{3214352} - q^3E_{3215342} - q^3E_{4321532}\\ &- q^3E_{5321432} + q^2E_{2352134} + q^2E_{2435213} + q^2E_{3243521} + q^2E_{5324321} - qE_{2343521} - qE_{2534321}\\ &- qE_{3254321} + (q^3 - q)E_{2123543} + (q^4 + q^2)E_{2134352} + (q^4 + q^2)E_{2135423}\\ &+ (q^4 + q^2)E_{2153432} + (q^4 + q^2)E_{3215432} + (-q^5 - q)E_{2135432} + E_{2354321},\end{aligned}$$

Central Element

Theorem

and,

$$
\begin{aligned}
A_{21} = & -q^3F_{12341235} + q^3F_{12354312} - q^3F_{12534123} + q^3F_{13212354} - q^3F_{13253412} + q^3F_{13543212} \\
& + q^3F_{14321235} - q^3F_{14325312} + q^3F_{15321234} + q^3F_{15432123} + q^3F_{21235431} - q^3F_{21352341} \\
& - q^3F_{21435231} - q^3F_{32143521} - q^3F_{53214321} - (q^4+q^2)F_{12235431} - (q^4+q^2)F_{11235432} \\
& + (q^4+q^2)F_{12352341} + (q^4+q^2)F_{12354123} + (q^4+q^2)F_{12435231} + (q^4+q^2)F_{13243521} \\
& + (q^4+q^2)F_{15324321} + (q^4+q^2)F_{21343521} + (q^4+q^2)F_{21534321} + (q^4+q^2)F_{32154321} \\
& - q(q^2+1)^2F_{12343521} - q(q^2+1)^2F_{12534321} - q(q^2+1)^2F_{13254321} \\
& - q(q^4+q^2+1)F_{21354321} + (q^6+q^4+q^2+1)F_{12354321} \\
A_{22} = & -q^3E_{12341235} + q^3E_{12354312} - q^3E_{12534123} + q^3E_{13212354} - q^3E_{13253412} + q^3E_{13543212} \\
& + q^3E_{14321235} - q^3E_{14325312} + q^3E_{15321234} + q^3E_{15432123} + q^3E_{21235431} - q^3E_{21352341} \\
& - q^3E_{21435231} - q^3E_{32143521} - q^3E_{53214321} - (q^4+q^2)E_{12235431} - (q^4+q^2)E_{11235432} \\
& + (q^4+q^2)E_{12352341} + (q^4+q^2)E_{12354123} + (q^4+q^2)E_{12435231} + (q^4+q^2)E_{13243521} \\
& + (q^4+q^2)E_{15324321} + (q^4+q^2)E_{21343521} + (q^4+q^2)E_{21534321} + (q^4+q^2)E_{32154321} \\
& - q(q^2+1)^2E_{12343521} - q(q^2+1)^2E_{12534321} - q(q^2+1)^2E_{13254321} \\
& - q(q^4+q^2+1)E_{21354321} + (q^6+q^4+q^2+1)E_{12354321},
\end{aligned}
$$

Central Element

Theorem

and,

$$
\begin{aligned}
A_{23} &= q^4F_{12354} - q^3F_{12435} - q^3F_{12534} - q^3F_{13254} - q^3F_{21354} + q^2F_{12543} + q^2F_{14325} + q^2F_{15324} \\
&\quad + q^2F_{21435} + q^2F_{21534} + q^2F_{32154} - qF_{15432} - qF_{21543} - qF_{43215} - qF_{53214} + F_{54321} \\
A_{24} &= q^4E_{45321} - q^3E_{43521} - q^3E_{45231} - q^3E_{45312} - q^3E_{53421} + q^2E_{34521} + q^2E_{42351} + q^2E_{43512} \\
&\quad + q^2E_{45123} + q^2E_{52341} + q^2E_{53412} - qE_{23451} - qE_{34512} - qE_{41235} - qE_{51234} + E_{12345} \\
A_{25} &= q^3F_{1235} - q^2F_{1253} - q^2F_{1325} - q^2F_{2135} + qF_{1532} + qF_{2153} + qF_{3215} - F_{5321} \\
A_{26} &= q^3E_{5321} - q^2E_{3521} - q^2E_{5231} - q^2E_{5312} + qE_{2351} + qE_{3512} + qE_{5123} - E_{1235} \\
A_{27} &= q^5F_{123543} - q^4F_{123435} - q^4F_{125343} - q^4F_{213543} + q^3F_{132435} + q^3F_{132534} + q^3F_{143253} + q^3F_{153243} \\
&\quad + q^3F_{213435} + q^3F_{215343} - q^2F_{134325} - q^2F_{153432} - q^2F_{321435} - q^2F_{321534} - q^2F_{432153} \\
&\quad - q^2F_{532143} + qF_{135432} + qF_{343215} + qF_{534321} + (q^3+q)F_{321543} + (-q^4-q^2)F_{132543} - F_{354321} \\
A_{28} &= q^5E_{354321} - q^4E_{354312} - (q^4-q^2)E_{343521} - (q^4-q^2)E_{534321} + q^3E_{342351} + q^3E_{352341} + q^3E_{435231} \\
&\quad + q^3E_{534231} - q^2E_{323541} - q^2E_{341235} - q^2E_{351234} - q^2E_{432351} - q^2E_{435123} - q^2E_{532341} \\
&\quad - q^2E_{534123} - q^2E_{543231} + qE_{235431} + qE_{312354} + qE_{431235} + qE_{531234} + qE_{543123} \\
&\quad + (q^3-q)E_{343512} + (q^3-q)E_{534312} + (q^3+q)E_{354123} + (-q^4-q^2)E_{354231} - E_{123543},
\end{aligned}
$$

Central Element

Theorem

and,

$$
\begin{aligned}
A_{29} = {} & q^6F_{1235432} - q^5F_{1234352} - q^5F_{1253432} - q^5F_{1325432} + q^4F_{1235234} + q^4F_{1243523} + q^4F_{1324352} \\
& + q^4F_{1532432} - (q^4 - q^2)F_{1223543} - q^3F_{1235423} - q^3F_{2134235} - q^3F_{2135234} - q^3F_{2143523} - q^3F_{2153423} \\
& - q^3F_{2354213} - q^3F_{3214352} - q^3F_{3215342} - q^3F_{4321532} - q^3F_{5321432} + q^2F_{2352134} + q^2F_{2435213} \\
& + q^2F_{3243521} + q^2F_{5324321} - qF_{2343521} - qF_{2534321} - qF_{3254321} + (q^3 - q)F_{2123543} + (q^4 + q^2)F_{2134352} \\
& + (q^4 + q^2)F_{2135423} + (q^4 + q^2)F_{2153432} + (q^4 + q^2)F_{3215432} + (-q^5 - q)F_{2135432} + F_{2354321} \\
A_{30} = {} & q^6E_{2354321} + q^4E_{2343512} + q^4E_{2534312} + (q^4 - q^2)E_{2352341} + (q^4 - q^2)E_{2435231} - (q^4 - q^2)E_{2235431} \\
& + (q^4 - q^2)E_{3243521} + (q^4 - q^2)E_{5324321} - q^3E_{2341235} - q^3E_{2351234} - q^3E_{2435123} - q^3E_{2534123} \\
& - q^3E_{3243512} - q^3E_{3253412} - q^3E_{4325312} - q^3E_{5324312} + q^2E_{2312354} + q^2E_{2431235} + q^2E_{2531234} \\
& + q^2E_{2543123} + q^2E_{3241235} + q^2E_{3251234} + q^2E_{3432512} + q^2E_{4325123} + q^2E_{5324123} + q^2E_{5343212} \\
& - qE_{2123543} - qE_{3212354} - qE_{3543212} - qE_{4321235} - qE_{5321234} - qE_{5432123} + (-q^3 - q)E_{3254123} \\
& + (q^4 + q^2)E_{2354123} + (q^4 + q^2)E_{3254312} + (-q^5 - q)E_{2354312} + (-q^5 + q)E_{2343521} + (-q^5 + q)E_{2534321} \\
& + (-q^5 + q)E_{3254321} + E_{1235432},
\end{aligned}
$$

Central Element

Theorem

and,

$$A_{31} = q^3 F_{4321} - q^2 F_{3421} - q^2 F_{4231} - q^2 F_{4312} + qF_{2341} + qF_{3412} + qF_{4123} - F_{1234}$$
$$A_{32} = q^3 E_{1234} - q^2 E_{1243} - q^2 E_{1324} - q^2 E_{2134} + qE_{1432} + qE_{2143} + qE_{3214} - E_{4321}.$$

This element acts as $q^{10} + q^6 + q^4 + q^2 + 2 + \frac{1}{q^2} + \frac{1}{q^4} + \frac{1}{q^6} + \frac{1}{q^{10}}$ *times the identity in the fundamental representation of* $\mathcal{U}_q(\mathfrak{so}_{10})$.

So It Goes...
From the Central Element to Type D ASEP

Coproduct of the central element

Let $C \in \mathcal{U}_q(\mathfrak{so}_{10})$ denote the central element.

Recall that $\Delta(C) \in \mathcal{U}_q(\mathfrak{so}_{10}) \otimes \mathcal{U}_q(\mathfrak{so}_{10})$. Also recall that in the fundamental representation, elements of $\mathcal{U}_q(\mathfrak{so}_{10})$ act as 10×10 matrices.

Thus, we can view $\Delta(C)$ as a 100×100 matrix. Call this H.

Structure of H

With the right choice of basis, H is a direct sum of one 10×10 block, forty 2×2 blocks, and ten 1×1 blocks:

10×10						
	2×2					
		$\ddots$				
			2×2			
				1×1		
					$\ddots$	
						1×1

Quantum Hamiltonian

All of the 1×1 blocks have entry

$$\Lambda = q^{12} + q^6 + q^4 + q^2 + 2 + \frac{1}{q^2} + \frac{1}{q^4} + \frac{1}{q^6} + \frac{1}{q^{12}}.$$

Define the **quantum Hamiltonian**

$$\hat{H} = H - \Lambda \cdot \mathrm{Id}.$$

From $\hat{H}$ to a Markov process

The method to get to a Markov process is from [CGRS16]:

1. Find eigenvectors $g_0, \ldots, g_k$ of $\hat{H}$ with eigenvalue 0, called the **ground state vectors**.
2. For each i, let G_i be the diagonal matrix given by the entries of g_i.
3. Define a matrix L_i by removing rows and columns from $G_i^{-1}\hat{H}G_i$ until all off-diagonal entries are non-negative.

Then each L_i is the generator of a Markov process!

1×1 block

The 1×1 blocks of $\hat{H}$ are all (0). Thus, the corresponding block in the generator matrix will be $L_{1\times 1} = (0)$.

2 × 2 block

The 2 × 2 blocks are

$$\begin{bmatrix} -q^{10}+2q^8-q^6-\frac{1}{q^8}+\frac{2}{q^{10}}-\frac{1}{q^{12}} & \frac{(q^2-1)^2(q^{18}+1)}{q^{11}} \\ \frac{(q^2-1)^2(q^{18}+1)}{q^{11}} & -q^{12}+2q^{10}-q^8-\frac{1}{q^6}+\frac{2}{q^8}-\frac{1}{q^{10}} \end{bmatrix},$$

which have eigenvector $\begin{pmatrix} q \\ 1 \end{pmatrix}$, and conjugating by the corresponding diagonal matrix $\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}$ results in

$$L_{2\times 2} = r^2 \begin{bmatrix} * & \frac{q^{1-2n}+q^{2n-1}}{q} \\ q(q^{1-2n}+q^{2n-1}) & * \end{bmatrix},$$

where the * entries are chosen so the rows sum to zero.

10×10 block

Let

$$B_1 = -q^5 + 3q^3 - 3q + \frac{1}{q} - \frac{2}{q^5} + \frac{4}{q^7} - \frac{2}{q^9},$$
$$B_2 = -2q^3 + 4q - \frac{2}{q} + \frac{1}{q^5} - \frac{3}{q^7} + \frac{3}{q^9} - \frac{1}{q^{11}},$$
$$B_3 = q^{10} - 2q^8 + q^6 - 2q^2 + 4 - \frac{2}{q^2} + \frac{1}{q^6} - \frac{2}{q^8} + \frac{1}{q^{10}}.$$

Then the 10×10 block is $U^T + D + U$, ...

10 × 10 block

... where

$$U = \begin{bmatrix} 0 & B_1 & qB_1 & q^2B_1 & q^3B_1 & B_3 & q^6B_1 & q^5B_1 & q^4B_1 & q^3B_1 \\ 0 & 0 & q^2B_1 & q^3B_1 & q^4B_1 & B_2 & B_3 & q^6B_1 & q^5B_1 & q^4B_1 \\ 0 & 0 & 0 & q^4B_1 & q^5B_1 & qB_2 & B_2 & B_3 & q^6B_1 & q^5B_1 \\ 0 & 0 & 0 & 0 & q^6B_1 & q^2B_3 & qB_2 & B_2 & B_3 & q^6B_1 \\ 0 & 0 & 0 & 0 & 0 & q^3B_3 & q^2B_2 & qB_2 & B_2 & B_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & q^6B_2 & q^5B_2 & q^4B_2 & q^3B_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^4B_2 & q^3B_2 & q^2B_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^2B_2 & qB_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

10×10 block

and D is the diagonal matrix with entries

$$\Big\{ -q^{10} + 2q^8 - q^6 - q^4 + 3q^2 - 3 + \frac{1}{q^2} - \frac{2}{q^6} + \frac{3}{q^8} - \frac{1}{q^{12}},$$

$$-q^{10} + 2q^8 - 2q^6 + 3q^4 - 3q^2 + 1 - \frac{2}{q^4} + \frac{4}{q^6} - \frac{3}{q^8} + \frac{2}{q^{10}} - \frac{1}{q^{12}},$$

$$-q^{10} + q^8 + 2q^6 - 3q^4 + q^2 - \frac{2}{q^2} + \frac{4}{q^4} - \frac{2}{q^6} - \frac{1}{q^8} + \frac{2}{q^{10}} - \frac{1}{q^{12}},$$

$$-2q^{10} + 5q^8 - 4q^6 + q^4 - 2 + \frac{4}{q^2} - \frac{2}{q^4} - \frac{1}{q^8} + \frac{2}{q^{10}} - \frac{1}{q^{12}},$$

$$-q^{12} + 2q^{10} - q^8 - 2q^2 + 4 - \frac{2}{q^2} - \frac{1}{q^8} + \frac{2}{q^{10}} - \frac{1}{q^{12}},$$

$$-q^{12} + 3q^8 - 2q^6 + q^2 - 3 + \frac{3}{q^2} - \frac{1}{q^4} - \frac{1}{q^6} + \frac{2}{q^8} - \frac{1}{q^{10}},$$

$$-q^{12} + 2q^{10} - 3q^8 + 4q^6 - 2q^4 + 1 - \frac{3}{q^2} + \frac{3}{q^4} - \frac{2}{q^6} + \frac{2}{q^8} - \frac{1}{q^{10}},$$

$$-q^{12} + 2q^{10} - q^8 - 2q^6 + 4q^4 - 2q^2 + \frac{1}{q^2} - \frac{3}{q^4} + \frac{2}{q^6} + \frac{1}{q^8} - \frac{1}{q^{10}},$$

$$-q^{12} + 2q^{10} - q^8 - 2q^4 + 4q^2 - 2 + \frac{1}{q^4} - \frac{4}{q^6} + \frac{5}{q^8} - \frac{2}{q^{10}},$$

$$-q^{12} + 2q^{10} - q^8 - 2q^2 + 4 - \frac{2}{q^2} - \frac{1}{q^8} + \frac{2}{q^{10}} - \frac{1}{q^{12}} \Big\},$$

10×10 block

This has four linearly independent eigenvectors

$$g_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -q^2 \\ q \\ 0 \\ 0 \\ 0 \\ -1 \\ q \end{pmatrix}, \quad g_1 = \begin{pmatrix} 0 \\ 0 \\ -q^2 \\ q \\ 0 \\ 0 \\ 0 \\ -1 \\ q \\ 0 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 \\ -q^2 \\ q \\ 0 \\ 0 \\ 0 \\ -1 \\ q \\ 0 \\ 0 \end{pmatrix}, \quad g_3 = \begin{pmatrix} -q^2 \\ q \\ 0 \\ 0 \\ 0 \\ -1 \\ q \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

These give four choices of ground state vector.

10×10 block

For each $\delta = 0, 1, 2, 3$, conjugate the Hamiltonian by G_δ, remove any rows containing ∞ and any columns containing 0 to obtain a 4×4 matrix L_δ. These are of the form

$$r^2 \begin{bmatrix} * & \frac{q^{2n-2} - q^{2n-4} + \frac{2}{q^2}}{q^{2\delta}} & \frac{\left(-q^{1-n} + q^{n-1}\right)^2}{q^2} & q^{2n-2} - q^{2n-4} + \frac{2}{q^2} \\ q^{-2\delta}\left(q^{2n} - q^{2n-2} + 2\right) & * & q^{-2n} - q^{2-2n} + 2 & \left(-q^{1-n} + q^{n-1}\right)^2 \\ q^2\left(-q^{1-n} + q^{n-1}\right)^2 & 2q^2 + q^{2-2n} - q^{4-2n} & * & q^{2\delta}\left(2q^2 + q^{2-2n} - q^{4-2n}\right) \\ q^{2n} - q^{2n-2} + 2 & \left(-q^{1-n} + q^{n-1}\right)^2 & q^{2\delta}\left(q^{-2n} - q^{2-2n} + 2\right) & * \end{bmatrix},$$

where the * entries are chosen so the rows sum to zero.

Combining the blocks

Taking the direct sum of one copy of L_δ, four copies of $L_{2\times2}$, and four copies of $L_{1\times1}$ yields in the generator matrix for the two-site Type D ASEP with parameters $(q, 5, \delta)$!

Duality: The Invisible String

Self-duality

Definition

Let $\mathcal{L}$ be the generator matrix of a Markov process M with state space $\mathfrak{X}$. Let $D : \mathfrak{X} \times \mathfrak{X} \to \mathbb{R}$ be a function. Let $\mathcal{D}$ be the matrix whose rows and columns are indexed by $\mathfrak{X}$ and whose (x, y) entry is $D(x, y)$.

If

$$\mathcal{L}\mathcal{D} = \mathcal{D}\mathcal{L}^T,$$

then M is **self-dual** with respect to the **duality function** D.

Additional definitions

Define the q-Pochhammer symbol for $a \in \mathbb{R}, m \in \mathbb{N}$ as

$$(a;q)_k := \prod_{i=0}^{k-1}(1-aq^i),$$

define the q-hypergeometric function ${}_2\varphi_1$ as

$${}_2\varphi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, z\right) := \sum_{k=0}^{\infty} \frac{(a;q)_k (b;q)_k}{(c;q)_k} \frac{z^k}{(q;q)_k},$$

and define the q-Krawtchouk polynomials as

$$K_n(q^{-x}; p, c; q) = {}_2\varphi_1\left(\begin{matrix} q^{-x}, q^{-n} \\ q^{-c} \end{matrix}; q, pq^{n+1}\right).$$

Motivation

We wish to prove the following theorem from [BBKUZ22] for general n:

Theorem

The Type D ASEP with $n = 2, 3$, $\delta = 0$ is self-dual with respect to the self-duality function

$$D^L_{\alpha_1,\alpha_2}(\eta,\xi) = D^L_{\alpha_1}(\eta_1,\xi_1) \cdot D^L_{\alpha_2}(\eta_2,\xi_2)$$

where

$$D^L_{\alpha_i}(\xi_i,\eta_i) = \prod_{x=1}^{L} K_{\eta_i^x}\left(q^{-2\xi_i^x}, p_i^x(\xi_i,\eta_i), 1, q^2\right)$$

and

$$p_i^x(\xi_i,\eta_i) = \alpha_i^{-1} q^{-2\left(N^-_{x-1}(\xi_i) - N^+_{x+1}(\eta_i)\right)+2x-2}.$$

Proof

The proof of the original theorem inducted on the number of lattice sites. The inductive step did not rely on n, so it remains to verify the base case of two sites.

We directly verified that $\mathcal{L}\mathcal{D} = \mathcal{D}\mathcal{L}^T$ using Python.

Would've, Could've, Should've (Future Directions)

Future directions

- Use the same process on $\mathcal{U}_q(\mathfrak{so}_{12})$.
- Generalize to $\mathcal{U}_q(\mathfrak{so}_{2n})$.
- Use the same process on $\mathcal{U}_q(\mathfrak{so}_6)$, but with a different representation.
- Apply this process to the exceptional Lie algebra of type E_6.
- Use the methods of [ZGB91], which uses universal R-matrices to find a different central element.

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