Localizations of Abelian Groups

Let $\mathfrak{C}$ be any category, A, B, C objects in $\mathfrak{C}$ and $\alpha : A \to B$ a morphism. The object C is called perpendicular to α, and we write $\alpha \perp C$, if for each morphism $\varphi : A \to C$ there is a unique $\psi : B \to C$ such that $\varphi = \psi \circ \alpha$. Now α is called a localization of A, if it happens that $\alpha \perp B$.

Under mild conditions on $\mathfrak{C}$, any localization α gives rise to a reflective subcategory $\alpha^\perp = \{C \in \mathfrak{C} : \alpha \perp C\}$ and an idempotent functor $\mathcal{L}_\alpha : \mathfrak{C} \to \alpha^\perp$ such that $\mathcal{L}_\alpha(A) = B$.

One of the best understood categories is the category $\mathfrak{Ab}$ of all abelian groups and we investigate localizations in that category. For example, it has been known for a long time that if $\eta : \mathbb{Z} \to H$ is a localization in $\mathfrak{Ab}$, then H is a commutative ring with identity such that all endomorphisms of the additive group of H are multiplications by elements of H. Such rings are known as E-rings and the functor $\mathcal{L}_\eta$ can be described. On the other hand, it turns out that for many subgroups L of $\mathbb{Q}$, the additive group of all rational numbers, there are localizations $\gamma : L \to M$, where not much is known about the functor $\mathcal{L}_\gamma$. Moreover, we show that localizations in $\mathfrak{Ab}$ exist in abundance.