Frontiers in Mathematics

Abstracts

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Singularities in Fluids

In this talk I will attempt to present some of the issues regarding singularities in incompressible fluids. I will give a few reasons why such issues might be important, describe a few model examples and review the recent history of the subject. The intention is to give a presentation that is suitable for a general audience, emphasizing ideas over technique.

An Eulerian-Lagrangian Approach to incompressible fluids

In these two lectures I will present is some detail a certain approach to viscous and inviscid fluids. The Euler equations can be written in an Eulerian-Lagrangian formulation – a formulation in Eulerian coordinates that describes the inverse of the Lagrangian particle trajectory map $x \mapsto A(x,t)$. This formulation is $(\partial_t + u \cdot \nabla) A = 0, \quad u = W[A]$ where $W[A]$ is the Weber formula. The Navier-Stokes equations also admit an Eulerian-Lagrangian formulation in terms of an appropriate diffusive map A and an additional field, the virtual velocity v. The term "virtual" refers to fields that, in the absence of viscosity, are time independent functions of the Lagrangian labels. Using the Eulerian-Lagrangian approach one can prove bounds that hold for all time for the diffusive map $A(x,t)$, its Eulerian gradient $\nabla A(x,t)$ and even its second derivatives $\nabla\nabla A(x,t)$. The Eulerian-Lagrangian approach affords a distinction between the stretching of Eulerian line elements (basically an inviscid process), and the viscous evolution of the virtual fields. Coefficients C involving second order derivatives of A arise when one computes the commutator between the Eulerian gradient and the Lagrangian gradient. These coefficients evolve in time, starting from zero, and enter as basic coefficients in the equations obeyed by virtual velocity and virtual vorticity. I will describe a Cauchy formula for the viscous Navier-Stokes equation that expresses the Eulerian vorticity in terms of the diffusive map A and the virtual vorticity, in the exact same manner as in the Euler equations. The difference is that the virtual vorticity is no longer a time independent function of A. For short times the virtual vorticity ζ decays. Long-lived Navier-Stokes solutions are represented as the many iterations of short time, near identity transformations. Other fluid models and approaches can be understood in this context.