Cracking the “Crack Problem” via meromorphic approximation

Abstract

We discuss some new links between approximation theory in the complex domain and a family of inverse problems for the 2D Laplacian related to nondestructive testing. More precisely, let $D$ be a bounded, simply connected domain with smooth boundary $\Gamma$ and suppose that $D$ contains an unknown one-dimensional crack $\gamma$ (a smooth Jordan arc with distinct endpoints $\gamma_0, \gamma_1$). We consider the Neumann boundary value problem:

$$
\Delta u = 0 \quad \text{in} \quad D \setminus \gamma,
\frac{\partial u}{\partial n_\Gamma} = \Phi \quad \text{on} \quad \Gamma,
\frac{\partial u}{\partial n_\gamma}^\pm = 0 \quad \text{on} \quad \gamma \setminus \{\gamma_0, \gamma_1\},
$$

where $n_\Gamma$ denotes the outer unit normal vector to $\gamma$, $n_\gamma$ one of the two unit normal vectors to $\gamma$, and $\Phi \in L^2(\Gamma)$ is, in the thermal framework, a prescribed heat flux along $\Gamma$. The inverse problem then consists of determining the existence and location of $\gamma$ from knowledge of the function $\Phi$ on $\Gamma$. In the special case when $D$ is the unit disk and the crack is a hyperbolic line segment, e.g. $[a, b] \subset (-1, 1)$, the method we describe concerns the asymptotic behavior of the poles of best meromorphic approximants on $\Gamma$ to a Markov function

$$
f(z) = \frac{1}{2\pi i} \int_a^b \frac{d\mu(t)}{t - z}.
$$

Connections with Hankel operators and the asymptotic behavior of singular numbers for these operators will also be discussed.

References
