Cracking the "Crack Problem" via meromorphic approximation

Abstract

We discuss some new links between approximation theory in the complex domain and a family of inverse problems for the 2D Laplacian related to nondestructive testing. More precisely, let D be a bounded, simply connected domain with smooth boundary Γ and suppose that D contains an unknown one-dimensional crack γ (a smooth Jordan arc with distinct endpoints γ_0, γ_1). We consider the Neumann boundary value problem:

$$\begin{aligned} \triangle u &= 0 \quad &\text{in} \quad & D \backslash \gamma, \\ \frac{\partial u}{\partial n_\Gamma} &= \Phi \quad &\text{on} \quad & \Gamma, \\ \frac{\partial u^\pm}{\partial n_\gamma} &= 0 \quad &\text{on} \quad & \gamma \backslash \{\gamma_0, \gamma_1\}, \end{aligned} \tag{$*$}$$

where n_Γ denotes the outer unit normal vector to γ, n_γ one of the two unit normal vectors to γ, and $\Phi \in L^2(\Gamma)$ is, in the thermal framework, a prescribed heat flux along Γ. The inverse problem then consists of determining the existence and location of γ from knowledge of the function Φ on Γ. In the special case when D is the unit disk and the crack is a hyperbolic line segment, e.g. $[a, b] \subset (-1, 1)$, the method we describe concerns the asymptotic behavior of the poles of best meromorphic approximants on Γ to a Markov function

$$f(z) = \frac{1}{2\pi i} \int_a^b \frac{d\mu(t)}{t - z}.$$

Connections with Hankel operators and the asymptotic behavior of singular numbers for these operators will also be discussed.

References

1. L. Baratchart, V. Prokhorov and E.B. Saff, "On Meromorphic Approximation of Markov Functions", manuscript.
2. L. Baratchart, J. Leblond, F. Mandrea and E.B. Saff, "How can meromorphic approximation help to solve some 2D inverse problems for the Laplacian?" Inverse Problems Journal, vol. 15 (1999), 79-90.