Families of sections of a convex body

It is a consequence of a result due independently to C. Bessaga (1958) and to B. Grünbaum (1958) that for every pair of positive integers $k, n$ with $1 < k < n$ there exists a finite family of centrally symmetric convex bodies in $\mathbb{R}^k$ whose affine images cannot be central sections of the same centrally symmetric convex body in $\mathbb{R}^n$. The smallest cardinality $N = N(k, n)$ of such a family is unknown. V. Klee (1960) asked, in a different language, what happens if one considers families of $k$-dimensional central sections of the unit ball of the space $\ell^p$ for $1 \leq p < \infty$. The answer turns out to depend on the arithmetic properties of $p$. It is related to the following well known problem: Given an even integer $p$ find the smallest positive integer $h = h(k, p)$ such that the unit ball of the space $\ell^p_h$ has a $k$-dimensional central section which is an ellipsoid.

The proofs of the results involve various techniques from topology, dimension theory, harmonic analysis, convexity, and local theory of Banach spaces.