Quantization in mathematics

Graduate Talk

1. Could mathematicians have anticipated quantization?

The most difficult obstacle to understanding quantum theory is that one must first abandon the most basic notion of classical mathematics: we have been using the "wrong" scalars. In 1925, Heisenberg discovered that the Newtonian equations of mechanics were correct provided one reinterpreted the "variables" as standing for self-adjoint matrices rather than real valued functions. Von Neumann was the first mathematician to provide a rigorous framework for this theory (1927). He was also the first to realize that if mathematics is to continue to be a mirror of reality, we must consider quantized versions of mathematics itself.

We will review some of the extraordinary physical manifestations that led to this revolution. We will then outline various attempts by mathematicians to make quantization seem natural and perhaps even inevitable. We will include a sketch of the first serious mathematical quantizations, namely integration theory by Murray and von Neumann, and topology by by Gelfand and Neumark.

Colloquium

2. Quantizing functional analysis

Quantization has been successfully achieved in many areas of mathematics, including portions of analysis, algebraic topology, differential geometry, algebra, probability theory, and most recently functional analysis. Just as in physics, one finds completely unexpected new phenomena in these new disciplines.

A key mathematical development of of the last half century has been the discovery that Banach spaces have a beautiful and intricate structure. The success of this program has in part rested upon using techniques from classical harmonic analysis and probability theory. In 1984 Zhong-Jin Ruan showed that linear spaces of bounded operators could be characterized by the underlying matrix norms. This discovery meant that one could consider a vast array of non-commutative analogues of "classical" Banach space theory. Perhaps the most gratifying aspect of this research is how the non-commutative analogues of harmonic analysis and probability theory continue to play a crucial role in this theory. In this new context algebraic techniques play a key role.

Colloquium

3. Freedom and liberty in operator spaces

The notion of independence is central to classical probability theory. The prototypical examples of independent random variables is obtained by considering products of measure spaces, or tensor products of the function algebras they generate. Intrigued by Kesten's theory of random walks on free groups and a subsequent paper by Avitzour, Voiculescu suggested that one could also consider a purely non-classical analogue by using free products of their algebras. This led to his discovery of a "free" analogue of the central limit theorem. He went on to show that one could construct freely independent random variables by considering large matrices with Gaussian entries, a technique pioneered by Wigner for quantum applications.

Voiculescu's theory provides a compelling illustration of the liberating influence of von Neumann's original idea: free products simply don't make sense in the classical commutative realm. We will sketch how Voiculescu's theory has been used in operator theory by researchers such as Pisier and Shlyakhtenko, and we will then briefly consider the "freeon" analogue of quantum field theory.