Counting Primes, Groups and Manifolds

Let x be a large real number. How many primes of size at most x are there? The Prime Number Theorem gives the asymptotic answer and the Riemann Hypothesis controls the error term.

If Γ is a finitely generated group, one can ask: how many subgroups has Γ of index at most x? This subject received considerable attention in the last two decades under the rubric "Subgroup Growth."

Let $X = G/K$ be a symmetric space when G is a semi-simple Lie group and K a maximal compact subgroup. How many Riemannian manifolds of volume at most x are covered by X?

In the lectures we will show that these seemingly unrelated questions are actually very much related. We will describe some old and some new results concerning these questions.

The three lectures will be related, but an effort will be made to keep them sufficiently independent of each other.

Lecture 1: *Primes and groups of polynomial growth*

We will start with surveying the classical results on counting primes and show how these results (together with other tools such as finite simple groups, algebraic groups and p-adic Lie groups) give a characterization of groups of polynomial subgroup growth.

Lecture 2: *Counting congruence subgroups*

Sharp estimates will be given for the number of congruence subgroups in arithmetic groups.

Lecture 3: *Subgroup growth of lattices in semisimple Lie groups and counting manifolds*

The subgroup growth rate of lattices Γ in semisimple groups G is shown to depend on G and not on Γ. The proof of this geometric fact depends, in the most general setting, on the generalized Riemann hypothesis, but in many cases, it is proved unconditionally. The number theory gives some geometric application s (some of them are somewhat unexpected).