Frontier Lectures February 28 - March 3

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1 Can one see the fundamental frequency of a drum?

In 1966 Marc Kac asked the famous question "Can one hear the shape of a drum?", formulating a goal to determine the shape of a domain in $\mathbb{R}^n$ by its frequencies i.e. eigenvalues of the Laplacian with the Dirichlet boundary conditions in this domain. This question stimulated thousands of papers in inverse spectral problems, which have numerous applications. The question in the title of this talk is somewhat opposite but seems very important too. The fundamental frequency of a domain in $\mathbb{R}^n$ is the lowest eigenvalue of the Dirichlet Laplacian in this domain, which plays an important role in mathematical physics and applications (e.g. to stability of mechanical systems). The goal is to find visual characteristics of the domain which determine or at least estimate the lowest eigenvalue. I will explain two-sided estimates for this eigenvalue in terms of capacity which is a characteristic of sets in $\mathbb{R}^n$ invented by M.Faraday and introduced to mathematics by N.Wiener.

2 Capacity in action

Capacity allows precise formulations of many results in analysis and partial differential equations which would be impossible without it. Important examples include the removable singularity theorem for harmonic functions (with an arbitrary singular set) and the Wiener criterion for regularity of a boundary point with respect to the Dirichlet boundary value problem for the Laplace equation. Another example is a Friedrichs' type inequality which relates L^2-norms of a function and its gradient in a ball, provided the function vanishes on a subset of this ball. These results will be explained in the lecture together with an important application in spectral theory.

3 Capacity and spectral properties of Schrödinger operators

I will explain how the capacity can be used in spectral theory of Schrödinger operators (possibly with magnetic field) to obtain necessary and sufficient conditions for some spectral properties (such as having purely discrete spectrum or being strictly positive), as well as two-sided estimates for the bottom of the spectrum. The results include a recent solution of a problem which was formulated by I.M.Gelfand in 1953. (This was done in a joint work by V.Maz'ya and M.Shubin, to appear in Annals of Mathematics.)