Texas A&M Frontier Lectures

Joseph Silverman (Brown University)

Lecture 1. Monday, April 4, 2005

A Number Theorist's Perspective on Dynamical Systems

The classical theory of discrete dynamical systems asks for a description of the behavior of points under iteration of a function,

$$F^n(z) = \underbrace{F \circ F \circ F \circ \cdots \circ F}_{n \text{ iterates}}(z).$$

The *orbit* of a point β is the set of images of β,

$$\operatorname{Orbit}(F, \beta) = \{\beta, F(\beta), F^2(\beta), F^3(\beta), \ldots\},$$

and the points with finite orbit, called the *preperiodic points*, play an especially important role in the dynamics of F.

Of particular interest are rational maps $F(z) = f(z)/g(z)$. For a number theorist, it is natural to take the polynomials $f(z)$ and $g(z)$ to have integer coefficients and to study the orbits of rational numbers β. Typical problems include:

(1) How many preperiodic points can be rational numbers?
(2) For which rational maps $F(z)$ can the orbit of a rational number contain infinitely many integers?
(3) What sorts of field extensions are generated by preperiodic points?
(4) What do orbits look like modulo p? or modulo p^n as $n \to \infty$?

In this initial talk I will briefly discuss some basic concepts from classical dynamics and then survey known results and outstanding conjectures in dynamics from a number theoretic perspective.

Lecture 2. Wednesday, April 6, 2005

Arithmetic Dynamics: Periodic Rationals and Wandering Integers

A rational map $F(z) = f(z)/g(z)$ has infinitely many complex *periodic points*, that is, points $\beta \in \mathbb{C}$ satisfying

$$F^n(\beta) = \beta \qquad \text{for some } n \geq 1.$$

Somewhat surprisingly, only finitely many of the periodic points can be rational numbers. In the first part of the talk I will sketch two proofs of this fact, one using the theory of height functions and the other based on reduction modulo p.

Periodic and preperiodic points are, in some sense, quite special, since the orbit for most starting points $\beta \in \mathbb{C}$ wanders around without ever repeating. If $F(z) \in \mathbb{Q}(z)$ and $\beta \in \mathbb{Q}$, then the orbit of β clearly consists entirely of rational numbers, but one may ask whether such a *wandering point* can visit infinitely many integers. In the second part of the talk I will answer this question and describe how it can be reduced to a problem in Diophantine approximation.

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Lecture 3. Thursday, April 7, 2005
Further Topics in Arithmetic Dynamics

In the final lecture of the series I will finish up anything left over from Lecture 2 and then discuss one or more of the following topics:

Canonical Heights. The canonical height associated to a rational map F is a real-valued function $\hat{h}_F$ with the property that

$$\hat{h}_F\bigl(F(\beta)\bigr) = (\deg F)\hat{h}_F(\beta).$$

The canonical height $\hat{h}_F(\beta)$ also provides a measure of the arithmetic complexity of the number β. I will explain how $\hat{h}_F$ is constructed and describe some of its useful properties.

***p*-adic and Nonarchimedean Dynamics**. Fields with an absolute value satisfying the nonarchimedean triangle inequality

$$|x+y| \le \max\bigl\{|x|, |y|\bigr\}$$

have many surprising properties. The dynamics of iteration of rational maps over such fields has many similarities and many striking differences with the classical theory over $\mathbb{C}$. I will survey some of the results known about nonarchimedean dynamical systems.

Moduli Spaces for Dynamics. There is a natural conjugation action of the group of linear fractional transformation $\left\{\frac{az+b}{cz+d}\right\}$ on the set of rational maps of degree D. The quotient space M_D is the natural moduli space for studying dynamical systems. I will describe the construction of M_D and some of the facts known about it geometry and its arithmetic.