Jet bundles, differential equations and hyperbolic algebraic varieties: I, II, III

Combined Abstract

The main goal of the talks will be to study complex varieties - mostly projective algebraic ones - through natural geometric properties related to hyperbolicity in the sense of Kobayashi. An important aspect is the strong connection between such geometric properties and the existence of global algebraic differential equations (foliations being a special case).

A convenient framework for this is the category of "directed manifolds", that is, the category of pairs (X, V) where X is a complex manifold and V a holomorphic subbundle of T_X. If X is compact, the pair (X, V) is hyperbolic if and only if there are no nonconstant entire holomorphic curves $f : \mathbf{C} \to X$ tangent to V (Brody's criterion). We will describe a construction of projectivized k-jet bundles P_kV, which generalizes a construction made by Semple in 1954 and allows to analyze hyperbolicity in terms of negativity properties of the curvature. More precisely, $\pi_k : P_kV \to X$ is a tower of projective bundles over X and carries a canonical line bundle $\mathcal{O}_{P_kV}(1)$; the hyperbolicity of X is then conjecturally equivalent to the existence of suitable singular hermitian metrics of negative curvature on $\mathcal{O}_{P_kV}(-1)$ for k large enough. The direct images $(\pi_k)_\star\mathcal{O}_{P_kV}(m)$ can be viewed as bundles of algebraic differential operators of order k and degree m, acting on germs of curves and invariant under reparametrization. Following an approach initiated by Green and Griffiths, we establish a fundamental vanishing theorem : every entire curve $f : \mathbf{C} \to (X, V)$ must satisfy any differential equation $P(f) = 0$ associated with a global jet differential taking values in a negative bundle.

In the first place, this leads in a natural way to the study of the algebra of (local) invariant differential operators. A second fundamental ingredient is the computation of global sections of jet differential bundles, e.g. via Riemann-Roch and ad hoc vanishing theorems. Major advances have been made in the last 10 years. In 1998, an important conjecture of Kobayashi on the hyperbolicity of generic surfaces of high degree in complex projective space $\mathbf{P}^3$ was settled independently by McQuillan and Demailly-El Goul, with respective bounds $d \geq 36$ and $d \geq 21$. New techniques developed since 2000 (Y.T.Siu, G.Dethloff-S.Lu, M.Paun, E.Rousseau, S.Diverio) produce similar results in higher dimensions and for more general varieties. We will discuss in particular a promising recent approach of S.Diverio relying on holomorphic Morse inequalities.