Geometric structures on uniruled projective manifolds:
varieties of minimal rational tangents

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Abstract. Let X be a uniruled projective manifold, i.e., a projective manifold that can be covered by rational curves. Fixing a polarization and minimizing degrees among free rational curves we have the notion of minimal rational curves. Let $\mathcal{K}$ be a minimal rational component, i.e., an irreducible component of the moduli space of minimal rational curves. Collecting all tangents to minimal rational curves passing through a given general point we obtain the variety of minimal rational tangents (VMRT). Let $x \in X$ be a general point and $\mathcal{U}_x$ be the moduli space of $\mathcal{K}$-curves marked at x. Then, there is a tangent map $\tau_x : \mathcal{U}_x \to \mathbb{P}T_x(X)$ which assigns to each element $[C] \in \mathcal{U}_x$ the tangent direction $[T_x(C)]$ at the given marking at x, and the VMRT $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ is the strict transform of $\mathcal{U}_x$ under the rational map τ_x. Together with Jun-Muk Hwang we have been studying uniruled projective manifolds in terms of the geometry of VMRTs. We put the emphasis on the case of Fano manifolds X of Picard number 1, the hard nuts according to Miyaoka.

Our general philosophy is to recover the structure of uniruled projective manifolds from their VMRTs. One of the key principles established in the geometric theory is the Cartan-Fubini Extension Principle, according to which a local biholomorphic map between two Fano manifolds of Picard number 1 extends automatically to a global biholomorphism under very mild conditions. For a uniruled projective manifold equipped with a minimal rational component $\mathcal{K}$, a basic object of study is the double fibration $\rho : \mathcal{U} \to \mathcal{K}$, $\mu : \mathcal{U} \to X$ given by the universal family of $\mathcal{K}$-curves. Exploiting the double fibration, associated distributions on $\mathcal{U}$ resp. $\mathcal{K}$ and the projective geometry of the VMRT at a general point x, we show that any VMRT-preserving local biholomorphic map $f : X \to X'$ between $(X, \mathcal{K})$ and $(X', \mathcal{K}')$ of Picard number 1 must map open pieces of minimal rational curves into minimal rational curves, provided that the Gauss map on $\mathcal{C}_x$ is generically injective. We then develop a method of analytic continuation to show that any such map must extend to a biholomorphism $F : X \to X'$. Together with the work of Kebekus on the tangent map and of Cho–Miyaoka–Shepherd-Barron on the characterization of projective spaces, we have extended our method to yield the Extension Principle whenever the general VMRT is not a finite union of projective linear subspaces.

In the special case of irreducible Hermitian symmetric spaces of the compact type of rank ≥ 2, the Extension Principle was known in the theory of G-structures by a Theorem of Ochiai's. For rational homogeneous spaces of Picard number 1 defined by long simple roots other than the projective space, this follows from the work of Yamaguchi on differential systems. From this perspective, the Cartan-Fubini Extension Priniciple underlies many of the works on rational homogeneous spaces undertaken by Hwang and myself. The general form of the Extension Principle has now been applied to give a solution of the Lazarsfeld Problem independent of Lie Theory, and it yields at the same time the rigidity of finite holomorphic maps onto most Fano manifolds of Picard number 1 when the domain manifold is fixed.