Rigidity of rational homogeneous spaces
of Picard number 1 under Kähler deformation
from case studies to general principles

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Abstract. By a theorem of Bott's any rational homogenenous space $S = G/P$ is infinitesimally rigid. As is well-known, $S := \mathbb{P}^1 \times \mathbb{P}^1$ can be deformed to any Hirzebruch surface of even genus. Such a jump phenomenon also occurs for $S := \mathbb{P}T_{\mathbb{P}^{2n+1}}$. In the latter case $S = G/P$ with G simple while S is of Picard number 2. It has been a folklore conjecture that any rational homogeneous space $S = G/P$ of Picard number 1 is rigid under deformation. So far the latter has only been established for projective spaces by Siu, and for hyperquadrics by Hwang. As a first step in our programme on VMRTs, we studied the more restricted problem of rigidity of such $S = G/P$ under Kähler deformation. We proved

Main Theorem [HM 2005]. *Let $S = G/P$ be a rational homogeneous space of Picard number 1. Let $\pi : \mathcal{X} \to \Delta := \{t \in \mathbb{C} : |t| < 1\}$ be a regular family of projective manifolds such that $X_t := \pi^{-1}(t)$ is biholomorphic to S for $t \neq 0$. Then, the central fiber X_0 is also biholomorphic to S.*

The problem was solved in a long series of articles. In retrospect it provided an ideal testing ground for schematizing the study of G/P of Picard number 1 from a geometric view-point. The classification theory of $S = G/P$ of Picard number 1 provides a hierarchy on the totality of such spaces in terms of their complexity: (1) irreducible Hermitian symmetric spaces of the compact type; (2) contact Fano manifolds of Picard number 1 other than odd-dimensional projective spaces; (3) other S defined by long simple roots in the Dynkin diagram, and (4) S defined by short simple root in the Dynkin diagram. We will explore the input into the general theory at each level of the hierarchy. The study of rigidity under deformation is made possible by the Kähler condition as the latter prevents splitting of minimal rational curves when passing to the limit in the central fiber, and the idea is to prove the invariance of VMRTs and to reconstruct X_0 from its VMRTs.

Geometrically there are three essential phases: (a) the symmetric case leading to *distributions* spanned by VMRTs, giving a criterion in general for their *integrability* in terms of *projective-geometric* properties of the VMRTs; (b) the long root (including contact) case requiring a generalization to *differential systems* generated by VMRTs, studied by means of generators and relations arising from pencils of rational curves; (c) the short root case leading to an obstruction theory for the degeneration of Lie algebras, as projective-geometric properties of the VMRT at a general point imposes serious restrictions on the degeneration of *holomorphic vector fields*. Finally, the Main Theorem would have been solved by identifying the VMRT at a general point of the central fiber with that of the model space, and by showing that the the model space can be *reconstructed from a single VMRT*. We are now able to prove the latter except for the short root case. The available proofs are however not uniform, as they rely on computations proving the vanishing of various forms of curvature.