## Solving the Tangential Cauchy-Riemann Equations: Old Paradigms and New Phenomena

The restriction of holomorphic functions of $n$ variables to appropriate submanifolds $M \subset \mathbb{C}^n$ satisfy a system of homogeneous first order linear partial differential equations called the tangential Cauchy-Riemann equations. The corresponding inhomogeneous system gives rise to what is called the $\overline{\partial}_b$-complex. Existence and regularity theorems for this system and the associated second order boundary Laplacian $\Box_b$ have important applications in complex analysis. The nature of these results depends strongly on the geometric properties of the manifold $M$. Optimal results are known in a number of important examples, but there are many open problems.

### Lecture 1 – Graduate Talk

### Fundamental solutions for the boundary Laplacian

The fundamental solution for $\Box_b$ on the boundary of a strictly pseudoconvex domain or a domain of finite type in $\mathbb{C}^2$ behaves much like a classical fractional integral operator, but with the Euclidean metric replaced by a single natural non-isotropic metric. Similarly, appropriate second derivatives of the fundamental solution behave like a singular integral operator. We will review some of this well-understood theory, and then mention some examples which fall outside this classical framework.

### Lecture 2 – Colloquium

### The boundary Laplacian on quadratic submanifolds of $\mathbb{C}^n$

We consider submanifolds $M \subset \mathbb{C}^n$ given by sets of quadratic equations. These manifolds naturally have the structure of nilpotent Lie groups, and the associated tangential Cauchy-Riemann equations are given by left-invariant vector fields. We can use representation theory to obtain $L^p$ regularity for fundamental solutions of the boundary Laplacian $\Box_b$. In contrast with the situations described in the first lecture, the fundamental solutions are not classical fractional integral operators, but seem best described in terms of distributions called *flag kernels* which are more closely related to product singular integral operators.

### Lecture 3 – Colloquium

### The boundary Laplacian on boundaries of decoupled domains

We study the $\overline{\partial}_b$-complex on boundary $M$ of a *decoupled domain*; this is a domain of the form $\left\{(z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} \,\middle|\, \Im\mathrm{m}[z_{n+1}] > P_1(z_1) + \cdots + P_n(z_n)\right\}$, where each $P_j$ is a subharmonic, non-harmonic polynomial. There is no group structure on $M$, but we can lift the problem to a submanifold of $\mathbb{C}^{2n}$ where again the fundamental solutions are given by product-like operators. A key point is that one need several non-equivalent metrics to describe the fundamental solution for $\Box_b$ on $M$.