Lectures on homogeneous varieties over finite fields

Given a system of polynomial equations over an arbitrary field K, one may first 'reduce' it to work over a finite field $\mathbf{F}_q$, and then count its solutions over any larger finite field $\mathbf{F}_{q^n}$. The resulting number of solutions, viewed as a function of n, satisfies deep regularity properties due to Weil and, in full generality, to Dwork, Grothendieck and Deligne.

The talks will address the special situation where the system defines a homogeneous variety X, ie, the solutions over the algebraic closure of K admit a transitive action of an algebraic group.

Lecture 1 will begin with classical examples and results on the number of points of homogeneous varieties over finite fields, and then turn to a recent result of Peyre and myself: if X is homogeneous under a linear algebraic group, then its counting function is a 'periodic polynomial' in q^n; moreover, the shifted periodic polynomial, where the variable t is replaced with $t+1$, has non-negative coefficients.

Lecture 2 will present an elementary proof of that result, based on some invariant theory of finite groups. A more advanced proof via cohomological methods will be outlined, and an analogous result for moduli of quiver representations (due to Mozgovoy and Reineke) will be discussed.

Lecture 3 will consider the case where X is homogeneous under an arbitrary group, e.g., an abelian variety. Then the counting function decomposes into a product of two functions associated with linear, resp. abelian 'factors' of X. This reflects a splitting property of the Albanese morphism over finite fields, somewhat analogous to the Borel–Remmert structure theorem for compact homogeneous Kähler manifolds. Some recent results suggested by this analogy will be presented.