HODGE THEORY FOR BEGINNERS

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LECTURE 1: Harmonic Representatives

LECTURE 2: Algebraic Cycles, Hodge Classes and some Commutative Algebra

LECTURE 3: Hodge Structures and Mumford-Tate Domains

LECTURE 1: Harmonic Representatives

LINEAR ALGEBRA

Basic objects of linear algebra:
Vector space V, linear transformation $L\colon V \to W$, basis, change of basis

Normal form: We can choose bases for V, W such that the matrix of L is in block form

$$L = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

The only invariant of L is the rank r
$\mathrm{Ker}(L) \subseteq V$, $\mathrm{Im}(L) \subseteq W$
Structure of L: $V \xrightarrow{\pi} V/\mathrm{Ker}(L) \xrightarrow{L_0} \mathrm{Im}(L) \xrightarrow{i} W$

π surjective, L_0 an isomorphism, i injective

When $L\colon V \to V$, the story is more complicated (Jordan normal form)

LINEAR GEOMETRY

Basic objects: Vector space V, positive-definite inner product $<,>$, linear transformation $L: V \to W$, orthonormal basis, orthogonal transformation

Normal form: Using orthonormal bases for V, W, in block form

$$L = \begin{pmatrix} \Lambda & 0 \\ 0 & 0 \end{pmatrix} \quad \textbf{singular value decomposition}$$

with Λ diagonal, diagonal entries $\lambda_1, \ldots, \lambda_r$ the **singular values** of L

Basic constructions:
(1) The **adjoint** of L: $L^*: W \to V$ defined by
$< Lv, w >=< v, L^*w >$ for all $v \in V$, $w \in W$
If we use orthonormal bases, the matrix of L^* is the transpose of the matrix of L

(2) If $S \subseteq V$ a linear subspace,
$S^\perp = \{v \in V \mid < v, s >= 0 \text{ for all } s \in S\}$, the **orthogonal complement** of S

Note $V = S \oplus S^\perp$, the **orthogonal direct sum decomposition**

(3) $\pi_S: V \to S$ **orthogonal projection**, $i_S: S \to V$ the **canonical inclusion**

Note: $\pi_S(v)$ is the point of S closest to v

RELATIONS BETWEEN THE BASIC CONSTRUCTIONS

$\mathrm{Ker}(L^*) = \mathrm{Im}(L)^\perp$

$\mathrm{Im}(L^*) = \mathrm{Ker}(L)^\perp$

$(L^*)^* = L$, $(ML)^* = L^* M^*$

$\pi_S^* = i_S$, $i_S^* = \pi_S$

Structure of L:

$V \xrightarrow{\pi} \mathrm{Ker}(L)^\perp \xrightarrow{L_0} \mathrm{Im}(L) \xrightarrow{i} W$

Now π is the orthogonal projection on $\mathrm{Ker}(L)^\perp$

$L^* = i_{\mathrm{Ker}(L)^\perp} \circ (L_0)^* \circ \pi_{\mathrm{Im}(L)}$

AN IMPORTANT TRICK

$\mathrm{Ker}(L^*L) = \mathrm{Ker}(L)$

$\mathrm{Im}(LL^*) = \mathrm{Im}(L)$

Proof: Clearly $\mathrm{Ker}(L) \subseteq \mathrm{Ker}(L^*L)$. Now if $L^*Lv = 0$, then $< L^*Lv, v >= 0$

But $0 =< L^*Lv, v >=< Lv, Lv >$, so $Lv = 0$ because $<, >$ is positive definite.
This gives the reverse containment. The result for Im follows by taking orthogonal subspaces.

Now

$V \cong \mathrm{Ker}(L) \oplus \mathrm{Ker}(L)^{\perp} = \mathrm{Ker}(L^*L) \oplus \mathrm{Im}(L^*L)$

$W \cong \mathrm{Ker}(LL^*) \oplus \mathrm{Im}(LL^*)$

GEOMETRIC INTERPRETATION

LL^*w is the point of $\text{Im}(L)$ closest to w

If $w \in \text{Im}(L)$, L^*w is the shortest solution of $Lv = w$

For any $w \in W$, the shortest $v \in V$ coming closest to solving $Lv = w$ is L^*w

EXAMPLE: RANKING SPORTS TEAMS

Set-up: G a directed graph with nodes Γ, edges E

$V = \text{Maps}(\Gamma, \mathbf{R})$
$W = \text{Maps}(E, \mathbf{R})$

$\Gamma = \{\text{Teams}\}$, $E = \{\text{Games}\}$
Edge $\vec{pq}$ denotes home team p, visiting team q

V = possible "skill levels" of teams
W = point spreads: visitors - home team

$L\colon V \to W$ maps $f \in V$ to $L(f)$ with
$L(f)(\vec{pq}) = f(q) - f(p)$
Interpretation: Point spread = skill level of q - skil level of p

$< f, g >= \sum_{p\in\Gamma} f(p)g(p)$ for $f, g \in V$
$< f, g >= \sum_{e\in E} f(e)g(e)$ for $f, g \in W$

If $w \in W$ is the observed point spread for the games played, then L^*w is one guess for the reconstructed skill levels of the teams

Comment: This is in the literature. There are many alternative methods–this one is pretty basic. The most principled way to do this would be to have a probabilistic model.

PROBABILISTIC INTERPRETATION OF L^*

Set-up: $w_{\text{true}} = Lv$
$w_{\text{obs}} = Lv + G$,
G is Gaussian noise, independent and with the same standard deviation on all variables

So $P(w_{\text{obs}} | w_{\text{true}}) = \text{const} \cdot e^{-\|w_{\text{obs}} - w_{\text{true}}\|^2 / 2\sigma^2}$

Then the maximum likelihood estimate for w_{true} is
$v_{\text{est}} = L^* w_{\text{obs}}$

Comment: If instead we put a Bayesian prior probability on v of
$\text{const} \cdot e^{-\|v\|^2 / 2\tau^2}$
then the maximum a posteriori estimate for v is

$v_{\text{est}} = (L^* L + (\sigma^2/\tau^2) I)^{-1} L^* w_{\text{obs}}$ **(Tikhonov regularization)**

HOMOLOGICAL ALGEBRA

Basic object: A **complex of vector spaces**
$(V^\bullet, L^\bullet)$ is $V^0 \xrightarrow{L^0} V^1 \xrightarrow{L^1} \cdots \xrightarrow{L^{n-1}} V^n$

with L^i linear transformations and key condition
$L^{i+1} \circ L^i = 0$ for all i, i.e.

$\mathrm{Im}(L^i) \subseteq \mathrm{Ker}(L^{i+1})$

Basic construction:

$H^k(V^\bullet) = \mathrm{Ker}(L^k)/\mathrm{Im}(L^{k-1})$,
the k'th **cohomology group** of the complex $V^\bullet$

GEOMETRIC HOMOLOGICAL ALGEBRA

If each V^k has a positive definite inner product $<,>$, then L^{k*} is defined

Important idea:

$\mathrm{Ker}(L^k) = \mathrm{Im}(L^{k-1}) \oplus (\mathrm{Im}(L^{k-1})^\perp \cap \mathrm{Ker}(L^k))$

So

$H^k(V^\bullet) \cong \mathrm{Im}(L^{k-1})^\perp \cap \mathrm{Ker}(L^k))$

But $\mathrm{Im}(L^{k-1})^\perp = \mathrm{Ker}(L^{k-1*})$

So $H^k(V^\bullet) \cong \mathrm{Ker}(L^k) \cap \mathrm{Ker}(L^{k-1*})$

THE LAPLACIAN IN HOMOLOGICAL ALGEBRA

Basic construction:

$\Delta^k = L^{k-1}L^{k-1*} + L^{k*}L^k$, the k'th **Laplacian**

Now $< \Delta^k v, v >=< L^{k-1}L^{k-1*}v, v > + < L^{k*}L^k v, v >$
$=< L^{k-1*}v, L^{k-1*}v > + < L^k v, L^k v >$

So $\Delta^k v = 0 \iff L^k v = 0$ and $L^{k-1*}v = 0$

Define $\mathcal{H}^k(V^\bullet) = \mathrm{Ker}(\Delta^k)$, the **harmonic cohomology** of $V^\bullet$

Thus $\mathcal{H}^k(V^\bullet) \cong H^k(V^\bullet)$

Since
$\mathrm{Ker}(\Delta^k) \subseteq \mathrm{Ker}(L^k)$,

Every cohomology class in $\mathrm{Ker}(L^k)/\mathrm{Im}(L^{k-1})$ has a unique **harmonic representative** in $\mathcal{H}^k(V^\bullet)$

HARMONIC, EXACT AND CO-EXACT

Often, we represent all of the L^k's in a complex by d, and the basic relation is

$d^2 = 0$

Now $\Delta_d = d^*d + dd^*$

We can write $V^k = \mathrm{Ker}(d) \oplus \mathrm{Ker}(d)^\perp$

Now $\mathrm{Ker}(d)^\perp = \mathrm{Im}(d^*)$, and

$\mathrm{Ker}(d) = (\mathrm{Ker}(d) \cap \mathrm{Ker}(d^*)) \oplus (\mathrm{Ker}(d) \cap \mathrm{Ker}(d^*)^\perp)$
$= \mathcal{H}^k \oplus (\mathrm{Ker}(d) \cap \mathrm{Im}(d))$

But $\mathrm{Im}(d) \subseteq \mathrm{Ker}(d)$, so

$\mathrm{Ker}(d) = \mathcal{H}^k \oplus \mathrm{Im}(d)$

Now $V^k = \mathrm{Ker}(d) \oplus \mathrm{Ker}(d)^\perp = \mathrm{Ker}(d) \oplus \mathrm{Im}(d^*)$

The final result is:

$V^k = \mathcal{H}^k \oplus \mathrm{Im}(d) \oplus \mathrm{Im}(d^*)$
or equivalently

$V^k = \mathcal{H}^k \oplus \mathrm{Im}(dd^*) \oplus \mathrm{Im}(d^*d)$

Note that these spaces are mutually orthogonal under $<,>$

THE DECOMPOSITION IN WORDS

Notation: $\text{Ker}(d)$ are the **closed** elements, $\text{Im}(d)$ are the **exact** elements, $\text{Ker}(d^*)$ are the **co-closed** elements, $\text{Im}(d^*)$ are the **co-exact** elements

(1) Every element decomposes uniquely into a sum of harmonic, exact and co-exact elements.
(2) Every closed element decomposes uniquely into a sum of harmonic and exact elements
(3) Every co-closed element decomposes uniquely into a sum of harmonic and co-exact elements
(4) An element is harmonic if and only if it is closed and co-closed

PROPERTIES OF Δ

$\Delta^* = \Delta$, i.e. Δ is **self-adjoint**

$< \Delta v, v > \geq 0$ for all v, with equality if and only if v is harmonic

THE LAPLACIAN WE KNOW AND LOVE

Note: So far, we have been dealing with finite-dimensional vector spaces, but a lot of this applies more generally.

$\mathcal{C}^\infty(\mathbf{R}) = \{\text{smooth functions on } \mathbf{R}\}$

$\mathcal{C}_c^\infty(\mathbf{R})$ denotes those with compact support, i.e. 0 outside of some finite interval $[-A, A]$ (A depends on the function)

Our complex is $\mathcal{C}^\infty(\mathbf{R}) \xrightarrow{d/dx} \mathcal{C}^\infty(\mathbf{R})$

$< f, g >= \int_{-\infty}^{\infty} f(x)g(x)\ dx = \int_{-A}^{A} f(x)g(x)dx$

$d = d/dx$

To compute d^*, we have by definition

$\int_{\mathbf{R}} f(dg/dx)\ dx = \int_{\mathbf{R}} (d^* f)g\ dx$

Doing integration by parts over an $[-A, A]$ that works for both f and g,

$\int_{-A}^{A} f(dg/dx)dx = fg|_{-A}^{A} - \int_{-A}^{A} (df/dx)g\ dx$

But $fg|_{-A}^{A} = 0$, so

$(d/dx)^* = -(d/dx)$, i.e. $d^* = -d$

At the left of the complex,

$\Delta = d^* d = -(d/dx) \circ (d/dx) = -d^2/dx^2$

This is the usual Laplacian, except for the minus sign (which geometers prefer)

FUNCTIONS AND VECTOR FIELDS ON $\mathbf{R}^3$

$\mathcal{C}^\infty(\mathbf{R}^3) = \{\text{smooth functions on } \mathbf{R}^3\}$
$\text{Vect}(\mathbf{R}^3) = \{\text{smooth vector fields on } \mathbf{R}^3\}$
$= \{M\hat{i} + N\hat{j} + P\hat{k} \mid M, N, P \in \mathcal{C}^\infty(\mathbf{R}^3)\}$

Recall from calculus:

$\text{Curl} \circ \vec{\nabla} = 0; \quad \text{Div} \circ \text{Curl} = 0$

So

$\mathcal{C}^\infty(\mathbf{R}^3) \xrightarrow{\vec{\nabla}} \text{Vect}(\mathbf{R}^3) \xrightarrow{\text{Curl}} \text{Vect}(\mathbf{R}^3) \xrightarrow{\text{Div}} \mathcal{C}^\infty(\mathbf{R}^3)$

is a complex

In order to define $<,>$, we use $\mathcal{C}_c^\infty(\mathbf{R}^3)$, $\text{Vect}_c(\mathbf{R}^3)$, the compactly supported functions and vector fields

$< f, g >= \int_{\mathbf{R}^3} fg \text{ dVol}$

$< \vec{X}, \vec{Y} >= \int_{\mathbf{R}^3} < \vec{X}, \vec{Y} > \text{dVol}$

ADJOINTS OF GRAD, CURL AND DIV

$< \vec{\nabla} f, M\hat{i} + N\hat{j} + P\hat{k} >= \int_{\mathbf{R}^3} (f_x M + f_y N + f_z P) \mathrm{dVol}$

If these have compact support, we can use the same integration by parts method to get this to equal

$- \int_{\mathbf{R}^3} (f M_x + f N_y + f P_z) \mathrm{dVol}$, and thus

$\vec{\nabla}^* = -\mathrm{Div}$

This of course, taking the adjoint of both sides, also gives

$\mathrm{Div}^* = -\vec{\nabla}$

A somewhat messier computation along the same lines shows

$\mathrm{Curl}^* = \mathrm{Curl}$

At the left of the complex,

$\Delta = \vec{\nabla}^* \vec{\nabla} = -\mathrm{Div} \circ \vec{\nabla} = -(\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2)$,

i.e. minus the usual Laplacian

At the first $\mathrm{Vect}(\mathbf{R}^3)$ in the complex, a messier computation shows

$(\mathrm{Curl}^* \mathrm{Curl} + \vec{\nabla} \vec{\nabla}^*)(M\hat{i} + N\hat{j} + P\hat{k})$ is minus the usual Laplacian, applied component by component.

Because there are no non-zero compactly supported harmonic functions (use the mean value property on a large sphere), we expect:

$\mathcal{C}_c^\infty(\mathbf{R}^3) = \mathrm{Div}(\mathrm{Vect}_c(\mathbf{R}^3)) = \mathrm{Div}\vec{\nabla}(\mathcal{C}_c^\infty(\mathbf{R}^3))$

$\mathrm{Vect}_c(\mathbf{R}^3) = \vec{\nabla}(\mathcal{C}_c^\infty(\mathbf{R}^3)) \oplus \mathrm{Curl}(\mathrm{Vect}_c(\mathbf{R}^3))$

Note: Of course, we need to use some elliptic operator theory to make all this work for smooth functions and vector fields–I have just concentrated on the formal side

FUNCTIONS AND VECTOR FIELDS ON THE TORUS

Instead, let's look at the torus

$T^3 = \mathbf{R}^3/\mathbf{Z}^3$

We may think of functions and vector fields as being triply periodic on $\mathbf{R}^3$

For functions on T^3,

$\mathrm{Ker}(\Delta) = \mathbf{R}$, i.e. harmonic functions on T are constants

The complex

$$\mathcal{C}^\infty(T^3) \xrightarrow{\vec{\nabla}} \mathrm{Vect}(T^3) \xrightarrow{\mathrm{Curl}} \mathrm{Vect}(T^3) \xrightarrow{\mathrm{Div}} \mathcal{C}^\infty(T^3)$$

has cohomology groups

$\mathbf{R}, \mathbf{R}^3, \mathbf{R}^3, \mathbf{R}$

Note that these are the same as the usual singular cohomology groups of the torus. This is an example of **De Rham's Theorem**

THE RIGHT WAY TO THINK ABOUT ALL THIS

For $\mathbf{R}^3$, let dx, dy, dz be symbols that **anti-commute**, under a multiplication denoted by $\wedge$, i.e. permuting two of them introduces a minus sign

$dy \wedge dx = -dx \wedge dy$

$dx \wedge dx = -dx \wedge dx$, forcing $dx \wedge dx = 0$

In the complex

$$\mathcal{C}^\infty(\mathbf{R}^3) \xrightarrow{\vec{\nabla}} \mathrm{Vect}(\mathbf{R}^3) \xrightarrow{\mathrm{Curl}} \mathrm{Vect}(\mathbf{R}^3) \xrightarrow{\mathrm{Div}} \mathcal{C}^\infty(\mathbf{R}^3)$$

we replace $M\hat{i} + N\hat{j} + P\hat{k}$ in the first $\mathrm{Vect}(\mathbf{R}^3)$ by

$M dx + N dy + P dz$, we call these 1-forms $A^1(\mathbf{R}^3)$

and in the second $\mathrm{Vect}(\mathbf{R}^3)$ by

$M dy \wedge dz + N dz \wedge dx + P dx \wedge dy$, we call these 2-forms $A^2(\mathbf{R}^3)$

We replace f in the $\mathcal{C}^\infty(\mathbf{R}^3)$ on the right by

$f dx \wedge dy \wedge dz$, we call these 3-forms $A^3(\mathbf{R}^3)$

and we leave f in the $\mathcal{C}^\infty(\mathbf{R}^3)$ on the left alone, we call these 0-forms $A^0(\mathbf{R}^3)$

THE EXTERIOR DERIVATIVE

We now define a linear map $d\colon A^k(\mathbf{R}^3) \to A^{k+1}(\mathbf{R}^3)$, the **exterior derivative**

For a function f, $df = f_x dx + f_y dy + f_z dz$

You did this in calculus, but this time we mean it

To take d of a k-form, we leave the dx, dy, dz alone, take d of each coefficient, and then collect terms using the rules for $\wedge$

$d(Mdx+Ndy+Pdz) = (M_x dx+M_y dy+M_z dz)\wedge dx+(N_x dx+ N_y dy + N_z dz) \wedge dy + (P_x dx + P_y dy + P_z dz) \wedge dz$
$= (P_y - N_z)dy \wedge dz + (M_z - P_x)dz \wedge dx + (N_x - M_y)dx \wedge dy$

Translating this back into vector fields, we see that

$d =$ Curl in this case

Similarly,

$d(Mdy\wedge dz+Ndz\wedge dx+Pdx\wedge dy) = M_x dx\wedge dy\wedge dz+N_y dy\wedge dz \wedge dz + P_z dz \wedge dx \wedge dy$
$= (M_x + N_y + P_z)dx \wedge dy \wedge dz$

Translating back into vector fields and functions, we see that

$d =$ Div in this case

Our complex is now

$A^0(\mathbf{R}^3)\xrightarrow{d}A^1(\mathbf{R}^3)\xrightarrow{d}A^2(\mathbf{R}^3)\xrightarrow{d}A^3(\mathbf{R}^3)$

called the **De Rham complex**

A BONUS: CROSS PRODUCT

Did you ever wonder where cross product came from, or why $\mathbf{R}^3$ has it but the other $\mathbf{R}^n$'s don't?

$(M_1 dx + N_1 dy + P_1 dz) \wedge (M_2 dx + N_2 dy + P_2 dz) = (N_1 P_2 - N_2 P_1) dy \wedge dz + (P_1 M_3 - P_3 M_1) dz \wedge dx + (M_1 N_2 - M_2 N_1) dx \wedge dy$

Translating 1-forms and 2-forms back into vector fields, we get that this is just cross product

We can do wedge product of 1-forms on any $\mathbf{R}^n$ and get a 2-form, but only on $\mathbf{R}^3$ and we identify back 2-forms with vector fields

DE RHAM COHOMOLOGY ON MANIFOLDS

A **smooth manifold** M of dimension n is a reasonable topological space for which any small open set looks like an open set in $\mathbf{R}^n$ (such an identification is called **local coordinates**) with the condition that any two sets of local coordinates near a given point are smooth functions of each other

If $x_1, x_2, \ldots, x_n$ are local coordinates, then a k-form locally looks like

$\omega = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} f_{i_1 i_2 \cdots i_k} dx_{i_1} \wedge dx_{i_2} \wedge \cdots dx_{i_k}$

This defines $A^k(M) = \{k\text{–forms on } M\}$

$d\colon A^k(M) \to A^{k+1}(M)$ is defined similarly to on $\mathbf{R}^3$, and
$d^2 = 0$

$H^k(A^\bullet(M)) = \{\text{closed forms on } M\}/\{\text{exact forms on } M\}$
is called the **De Rham cohomology** of M, denoted
$H^k_{DR}(M)$

De Rham's Theorem says that $H^k_{DR}(M)$ is canonically isomorphic to the singular cohomology $H^k_{\text{sing}}(M, \mathbf{R})$

MAXWELL'S EQUATIONS

$\vec{E} = E_1\hat{i} + E_2\hat{j} + E_3\hat{k}$ the electric field

$\vec{B} = B_1\hat{i} + B_2\hat{j} + B_3\hat{k}$ the magnetic field

Maxwell's Equations
(1) $\vec{\nabla} \cdot \vec{B} = 0$
(2) $\vec{\nabla} \cdot \vec{E} = \rho$
(3) $\vec{\nabla} \times \vec{B} = \vec{J} + \partial\vec{E}/\partial t$
(4) $\vec{\nabla} \times \vec{E} = -\partial\vec{B}/\partial t$

The relativistic way to think of this is on space-time $\mathbf{R}^4$ with coordinates x, y, z, t

The **electromagnetic 2-form** is

$\Omega = B_1 dy \wedge dz + B_2 dz \wedge dx + B_3 dx \wedge dy + E_1 dx \wedge dt + E_2 dy \wedge dt + E_3 dz \wedge dt$

Amazingly,

$d\Omega = 0 \iff (1), (4)$

What about (2), (3)?

THE OTHER MAXWELL'S EQUATIONS

Special relativity is built on the **Minkowski distance** on $\mathbf{R}^4$

Distance from (x_1, y_1, z_1, t_1) to (x_2, y_2, z_2, t_2) is

$((x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 - (t_1 - t_2)^2)^{(1/2)}$

where we take the speed of light to be 1

Notice the minus sign before the $(t_1 - t_2)^2$ term

We get an inner product on the space spanned by dx,dy,dz,dt:

$(dx)^2 + (dy)^2 + (dz)^2 - (dt)^2$

For example, for a 1-form $\omega = Adx + Bdy + Cdz + Ddt$ with compact support,

$< \omega, \omega >= \int_{\mathbf{R}^4} (A^2 + B^2 + C^2 - D^2)\text{dVol}$

and for a 2-form

$\omega = A_1 dy \wedge dz + A_2 dz \wedge dx + A_3 dx \wedge dy + B_1 dx \wedge dt + B_2 dy \wedge dt + B_3 dz \wedge dt$,

$< \omega, \omega >= \int_{\mathbf{R}^4} (A_1^2 + A_2^2 + A_3^2 - B_1^2 - B_2^2 - B_3^2)\text{dVol}$

Using this inner product, we can formally define d^* using integration by parts. If we do this, the other Maxwell equations are:

$d^* \Omega = J$, where J is a 1-form whose four components incorporate $\vec{J}$ and ρ

MAXWELL'S EQUATIONS AS A WAVE EQUATION

(1) $d\Omega = 0$

(2) $d^*\Omega = J$

Now

$H^2(A^\bullet(\mathbf{R}^4)) = 0$, so we can write

$\Omega = dA$ for some 1-form A, called the **vector potential**

In fact, since $\mathrm{Im}(d) = \mathrm{Im}(dd^*)$, we can write

$\Omega = dd^*\Phi$ for some 2-form Φ. Now if we take $A = d^*\Phi$, then $d^*A = (d^*)^2\Phi = 0$

So (2) can be rewritten as

$d^*dA = J$, or more suggestively

$(dd^* + d^*d)A = J$

Now writing $dd^* + d^*d$ as Δ, but remembering we are using the Minkowski inner product, we get

$\Delta A = J$, where Δ is, component by component, the operator

$\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2 - \partial^2/\partial t^2$

Comment: This is all formal. We no longer have elliptic operators.

More interesting comment: If the region we are working on has non-trivial topology, we cannot define A globally on the region. There is a famous physics experiment, the **Aharonov-Bohm effect**, that shows this actually matters in the real world.

MAXWELL'S EQUATIONS IN A VACUUM

(1) $d\Omega = 0$

(2) $d^*\Omega = 0$

or equivalently

Ω is harmonic

TANGENT SPACE TO A MANIFOLD

M smooth manifold of dimension n, and $p \in M$

Informal definition: $T_p(M) =$ usual tangent space to $\mathbf{R}^n$ using local coordinates

More elegant but more opaque definition: $\mathcal{I}_p$ =smooth functions on a neighborhood of p vanishing at p

$T_p(M) =$ dual space of $(\mathcal{I}_p/\mathcal{I}_p^2)$

Analyst's definition: $T_p(M) =$ first-order linear differential operators on M at p

RIEMANNIAN MANIFOLDS

Given a smooth manifold M, it is possible to define a distance by putting a positive definite inner product on the tangent space to M at each point–this is called a **Riemannian metric**

We choose local coordinates $x_1, x_2, \ldots, x_n$ so at a given point of M, $dx_1, \ldots, dx_n$ is an orthonormal basis

This lets us, by taking sums of squares of coefficients, define at every point a positive definite inner product on k-forms for every k

If we have a compact, oriented (not a Klein bottle!) manifold, we can integrate this against dVol to get $< \omega_1, \omega_2 >$, a positive definite inner product on $A^k(M)$

Now we can define d^* and $\Delta = dd^* + d^*d$ and $\mathcal{H}^k(M)$

Hodge, inspired by Maxwell's equations in the vacuum, defined harmonic forms by

$d\omega = 0$, $d^*\omega = 0$

This is a stellar example of the interplay between applications and pure mathematics.

Using elliptic operator theory, he proved

The Hodge Theorem: Every class in $H^k_{DR}(M)$ is represented by a unique harmonic form

COR: $\mathcal{H}^k(M) \cong H^k_{DR}(M) \cong H^k_{\text{sing}}(M, \mathbf{R})$

Comment: The question is–what is the payoff for doing this? If we know more about the geometry of M, can we say more? We can. But the payoff is especially great if we have a complex manifold. Stay tuned.

LECTURE 2: Algebraic Cycles, Hodge Classes and some Commutative Algebra

HYPERSURFACES OF DEGREE d

Set-up: A **homogeneous polynomial** $F(z_1, z_2, \ldots, z_n)$ of degree d is a polynomial all of whose monomials have total degree d, e.g $z_1^3 + z_2 z_3^2$ is homogeneous, but $z_1^3 + z_2 z_3$ is not. Alternatively,

$F(\lambda z_1, \ldots, \lambda z_n) = \lambda^d F(z_1, \ldots, z_n)$ for all $\lambda \in \mathbf{C}^*$

Complex projective space $\mathbf{CP}^n$ is
$\mathbf{CP}^n = \{\text{lines through } 0 \text{ in } \mathbf{C}^{n+1}\} = \mathbf{C}^{n+1} - \{\vec{0}\}/\sim,$

where $(z_1, \ldots, z_{n+1}) \sim (\lambda z_1, \ldots, \lambda z_{n+1})$ for $\lambda \in \mathbf{C}^*$

Note that for a homogeneous polynomial F, while F is not a function on $\mathbf{CP}^n$, $X = \{F = 0\}$ defines a subset of $\mathbf{CP}^n$. X is called a **hypersurface of degree** d or an $(n-1)$**-fold of degree** d

X is **smooth** if $\vec{\nabla} F = (\partial F/\partial z_1, \ldots, \partial F/\partial z_{n+1})$ is never 0 except at $(0, 0, \ldots, 0)$

LINES ON SURFACES

A **line** L in $\mathbf{CP}^n$ is defined by $n-1$ independent linear equations

We will be discussing smooth surfaces X of degree d in $\mathbf{CP}^3$ ask which ones contain a line

$d = 1$ A plane contains a 2-parameter family of lines

$d = 2$ A smooth quadric surface contains two 1-parameter families of lines

$d = 3$ A smooth cubic surface contains exactly 27 lines (this is a famous theorem)

$d = 4$ It is one condition for a smooth surface of degree 4 to contain a line

CONDITIONS TO CONTAIN A LINE

The homogeneous polynomials of degree d on $\mathbf{CP}^1$ have basis $z_1^d, z_1^{d-1}z_2, \ldots, z_2^d$, so the dimension is $d+1$

A hypersurface $X = \{F = 0\}$ contains a line L if and only if the restriction $F|_L = 0$

It is therefore $d+1$ conditions for a hypersurface X of degree d to contain a given line L

How many lines are there in $\mathbf{CP}^3$? A line in determined by two distinct points p_1, p_2, but we can slide each point along the line. So the dimension of {lines in $\mathbf{CP}^3$} is

3+3-1-1 =4

Alternatively, the line is defined by 2 equations, hence a matrix

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{pmatrix}$$

We can usually by Gaussian elimination change basis to

$$\begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{pmatrix}$$

This has 4 free entries, hence dimension 4. The other Gaussian elimination possibilities have lower dimension, e.g.

$$\begin{pmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix}$$

which has dimension 3. These possibilities lead to **Schubert cells**. Colleen Robles is an expert on much more sophisticated versions of these

DIMENSION COUNT TO CONTAIN A LINE

There are $d+1$ conditions to contain a given line L

There is a 4-dimensional family of lines in $\mathbf{PC}^3$

It is therefore

$(d+1)-4 = d-3$

conditions for a surface of degree d to contain *some* line

Notice that this fits all of the examples given before

It turns out that this illustrates a much more general phenomenon

ALGEBRAIC CURVES

A **hyperplane** in $\mathbf{CP}^n$ is just a hypersurface of degree 1, it is isomorphic to $\mathbf{CP}^{n-1}$

An **algebraic curve** C in $\mathbf{CP}^n$ is defined by homogeneous polynomials

$C = \{F_1 = 0, \ldots, F_r = 0\}$, where we decide it is a curve if its intersection with almost all hyperplanes H is a collection of points. The number of points is the same for almost all H, and this is called the **degree** of C

A line, of course, has degree 1

As you may suspect, I am glossing over some subtleties here, notably issues of multiplicity–the analogue of an equation having a multiple root. The part that is not intuitive is that the number r of equations we need may be larger than $n-1$

Example: C is defined by the 2×2 minors of the matrix

$$\begin{pmatrix} z_1 & z_2 & z_3 \\ z_2 & z_3 & z_4 \end{pmatrix}$$

This is called the **twisted cubic**, which has degree 3

A particularly simple type of curve contained in a surface X in $\mathbf{CP}^3$ is a **complete intersection curve on** X, $C = X \cap X'$, where X' is a surface of deg d'

Bezout's Theorem (a special case) A complete intersection C has degree dd'

These are the curves that any X has. Notice that a line, although on its own it is the intersection of two planes, can never be a complete intersection curve on X unless X is a plane

CONDITIONS TO CONTAIN OTHER CURVES

The restriction of homogeneous polynomials of degree d to a **plane conic** is a space of dimension

$(d+2)(d+1)/2 - d(d-1)/2 = 2d+1$

There is a 3-dimensional set of planes in $\mathbf{CP}^3$, and on each plane, a 5-dimensional family of plane conics

So it is $2d+1-(5+3) = 2d-7$ conditions on a hypersurface of degree d to contain some plane conic

The restriction of homogeneous polynomials of degree d to a twisted cubic is a space f dimension $3d+1$

There is a 12-dimensional family of twisted cubics in $\mathbf{CP}^3$

So it is $3d+1-12 = 3d-11$ conditions for a hypersurface of degree d to contain some twisted cubic

MORAL: We expect that more complicated curves impose more conditions

Like all morals, this is not strictly true, but for a fixed curve it becomes true for $d >> 0$.

Notice in our examples that all these curves give 1 condition for $d = 4$

THE NOETHER-LEFSCHETZ THEOREM

Noether-Lefschetz Theorem: For almost all surfaces X of degree $d \geq 4$, the only algebraic curves on X are complete intersection curves $X \cap X'$

Noether's strategy for proving this was to look at increasingly complicated types of algebraic curves, and to show that they imposed a progressively larger number of conditions

Unfortunately, there are a countable number of different types of algebraic curves, and so this strategy never realy worked

Lefschetz, as he understood better the topology of algebraic varieties, was able to give a proof using the **monodromy group**, which tracks how cohomology groups of surfaces of degree d fit together in a family

We will also take a roundabout approach going via the topology of algebraic varieties, as a way of illustrating the power of Hodge theory

As a bonus, we will obtain the:

Explicit Noether-Lefschetz Theorem [G]: Containing a non-complete-intersection algebraic curve imposes at least $d-3$ conditions on a smooth surface of degree d. (= holds in the case of lines, and for $d \geq 5$, this is the only type of curve for which we get equality–but we won't show this)

COMPLEX MANIFOLDS

A **complex manifold** M of dimension n has a reasonable topology and locally has complex local coordinates $z_1, \ldots, z_n$ mapping that neighborhood to an open set in $\mathbf{C}^n$.

Another set of complex local coordinates $w_1, \ldots, w_n$ overlapping $z_1, \ldots, z_n$ must be related to it by (on the overlap) an analytically invertible transformation

$$w_1 = f_1(z_1, \ldots, z_n), \ldots, w_n = f_n(z_1, \ldots, z_n)$$

where the f_i are analytic functions, i.e locally convergent power series in $z_1, \ldots, z_n$

If $z_j = x_j + iy_j$ is the decomposition of the local coordinates into real and imaginary parts, then

$$dz_j = dx_j + idy_j,\ d\bar{z}_j = dx_j - idy_j$$

There are also partial derivatives

$$\partial/\partial z_j = \partial/\partial x_j - i\partial/\partial y_j$$

$$\partial/\partial \bar{z}_j = \partial/\partial x_j + i\partial/\partial y_j$$

A TALE OF TWO BASES

There are two bases over **C** for the 1-forms:

$dx_1, \dots, dx_n, dy_1, \dots, dy_n$

and

$dz_1, \dots, dz_n, d\bar{z}_1, \dots, d\bar{z}_n$

For k forms, bases look like

$dx_{i_1} \wedge \cdots \wedge dx_{i_p} \wedge dy_{j_1} \wedge \cdots \wedge dy_{j_q}$

where $p + q = k$, and

$dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$

where $p + q = k$. Linear combinations of the latter, with function coefficients and p, q fixed, are called (p, q)**-forms**

THE CAUCHY-RIEMANN EQUATIONS: ELEGANT VERSION

$f(z) = u + iv$

$\partial f/\partial \bar{z} = (\partial/\partial x + i\partial/\partial y)f$

$= (u_x + iu_y) + i(v_x + iv_y)$

$= (u_x - v_y) + i(u_y + v_x)$

So $\partial f/\partial \bar{z} = 0 \Leftrightarrow u_x = v_y$ and $u_y = -v_x$

THE OPERATORS ∂ and $\bar{\partial}$

The Cauchy-Riemann equations are equivalent to saying:

$f(z_1, \ldots, z_n)$ is analytic $\Longleftrightarrow \partial f/\partial \bar{z}_j = 0$ for all j

There are now two operators taking functions to 1-forms:

$\partial f = \sum_j (\partial f/\partial z_j) dz_j$

$\bar{\partial} f = \sum_j (\partial f/\partial \bar{z}_j) d\bar{z}_j$

We can extend these to differential forms by acting on the coefficients and leave the dz_j and $d\bar{z}_j$'s alone

When we do this, we get:

$d = \partial + \bar{\partial}$, and

(1) $\partial^2 = 0$

(2) $\partial\bar{\partial} = -\bar{\partial}\partial$

(3) $\bar{\partial}^2 = 0$

Note that f is analytic if and only if $\bar{\partial} f = 0$

(p, q)-FORMS

Instead of the local basis $dx_1, \ldots, dx_n, dy_1, \ldots, dy_n$ for the 1-forms, we can use $dz_1, \ldots, dz_n, d\bar{z}_1, \ldots, d\bar{z}_n$

For 2-forms, a basis then looks like things of the form:

$dz_i \wedge dz_j$: forms of type $(2, 0)$

$dz_i \wedge d\bar{z}_j$: forms of type $(1, 1)$

$d\bar{z}_i \wedge d\bar{z}_j$: forms of type $(0, 2)$

In general, a local basis for k-forms decomposes into those with p dz's and q $d\bar{z}$'s with $p + q = k$

These are called **forms of type** (p, q), and the set of smooth global (p, q)-forms is denoted $A^{p,q}(M)$

Thus $A^k(M, \mathbf{C})) = \bigoplus_{p+q=k} A^{p,q}(M)$

The $\mathbf{C}$ is there to indicate that the coefficients are complex-valued functions

Note

$\partial\colon A^{p,q}(M) \to A^{p+1,q}(M)$

$\bar{\partial}\colon A^{p,q}(M) \to A^{p,q+1}(M)$

A MOMENTARY DISAPPOINTMENT

One might reasonably hope that for a complex manifold: $H^k_{DR}(M, \mathbf{C})$ would decompose into a sum of cohomology using (p, q) forms for $p + q = k$

Equivalently, a d-closed k-form ω would decompose into the sum of d-closed (p, q)-forms for $p + q = k$, plus perhaps a d-exact form

Unfortunately, this doesn't happen

However, it does happen for something called a **Kähler manifold**. Fortunately, smooth hypersurfaces, and actually all smooth algebraic varieties that are subvarieties of some $\mathbf{CP}^n$–these are called **projective varieties**–are Kähler manifolds

RIEMANNIAN AND HERMITIAN METRICS

Riemannian metrics are modeled on the inner product on $\mathbf{R}^n$:

$$< v, w >= \sum_j v_j w_j$$

On a complex manifold, a **Hermitian metric** is a Hermitian positive definite inner product modeled on the inner product on $\mathbf{C}^n$:

$$< v, w >= \sum_j v_j \bar{w}_j$$

Now for a Riemannian metric, we can always choose local coordinates $x_1, \ldots, x_n$ centered on the point p, i.e. $x_j(p) = 0$ for all j, so that, for 1-forms

$$< dx_i, dx_j >= \delta_{ij} + O(\|x\|^2)$$

These are called **geodesic local coordinates** at p

For a complex manifold with a Hermitian metric, we would like to find local complex coordinates $z_1, \ldots, z_n$ centered at p such that

$$< dz_i, dz_j >= \delta_{ij} + O(\|z\|^2)$$

This is not always possible, but when it is, we say that the Hermitian metric is a **Kähler metric**

WHAT'S SO GREAT ABOUT KÄHLER MANIFOLDS?

Once we have a compact complex manifold with a Hermitian metric, we can define d^*, ∂^*, $\bar{\partial}^*$ and

$$\Delta_d = dd^* + d^*d$$

$$\Delta_\partial = \partial\partial^* + \partial^*\partial$$

$$\Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$$

Now Δ_d may not preserve the (p, q) type, but by definition

$$\Delta_\partial \colon A^{p,q}(M) \to A^{p,q}(M)$$

$$\Delta_{\bar{\partial}} \colon A^{p,q}(M) \to A^{p,q}(M)$$

For Kähler manifolds, we have the wonderful identities:

$$\Delta_\partial = \Delta_{\bar{\partial}} = (1/2)\Delta_d$$

HARMONIC (p, q)-FORMS AND THE HODGE DECOMPOSITION

Let $\mathcal{H}^{p,q}(M) = \mathrm{Ker}(\Delta_{\bar{\partial}} \colon A^{p,q}(M) \to A^{p,q}(M)) =$

$\{\omega \in A^{p,q}(M) \mid \bar{\partial}\omega = 0, \bar{\partial}^*\omega = 0\} =$

$\{\omega \in A^{p,q}(M) \mid \partial\omega = 0, \partial^*\omega = 0\}$

Since $d = \partial + \bar{\partial}$, we see that forms in $\mathcal{H}^{p,q}$ are d-closed

This has the really great consequence that:

(1) $\mathcal{H}^{p,q}(M) \subseteq \mathcal{H}^{p+q}(M)$

(2) $\mathcal{H}^{k}(M) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(M)$ (the **Hodge decomposition**)

Although the harmonic spaces depend on the choice of Kähler metric, letting

$H^{p,q}(M) = \{\bar{\partial}-\text{closed } (p, q)-\text{forms}\}/\{\bar{\partial}-\text{exact } (p, q)-\text{forms}\}$ the decomposition

$H^k_{DR}(M, \mathbf{C}) = \bigoplus_{p+q=k} H^{p,q}(M)$ is independent of this

This is called the **Hodge decomposition**

INTEGRATION OF DIFFERENTIAL FORMS

On an n-dimensional real manifold M, then if M is compact and oriented, and $\omega \in A^n(M)$,

$\int_M \omega$ makes sense

Locally, using properly oriented local coordinates $x_1, \ldots, x_n$,

$\omega = f(x_1, \ldots, x_n) dx_1 \wedge \cdots \wedge dx_n$ and we can do the usual

$\int_U f(x_1, \ldots, dx_n) dx_1 dx_2 \cdots dx_n$

over a U contained in the coordinate patch we are working in.

Differential forms were invented to do integration, and they transform in just the right way that the answer is independent of the choice of oriented local coordinates

Now we add up the results of these local calculations over the entire manifold

If N is a k-dimensional submanifold of M or even a topological k-chain, and $\omega \in A^k(M)$, we can use a similar method to make sense of

$\int_N \omega$

POINCARÉ DUAL FORM

If N is a k-dimensional submanifold (or topological k-cycle) of an n-dimensional compact oriented manifold M, there exists

$\eta_N \in A^{n-k}(M)$ with $d\eta_N = 0$ such that

For all $\omega \in A^k(M)$ with $d\omega = 0$,

$\int_N \omega = \int_M \omega \wedge \eta_N$

η_N is called a **Poincaré dual form** for N, and

$[\eta_N] \in H^{n-k}_{DR}(M)$

is called the **Poincaré dual class** of N

INTEGRATION: COMPLEX CASE

For a compact complex manifold M of dimension n, then as a real manifold it has dimension $2n$, it is always orientable with a canonical orientation, and $\omega \in A^{2n}(M, \mathbf{C})$ in local coordinates will look like

$$\omega = f(z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n) dz_1 \wedge \cdots dz_n \wedge d\bar{z}_1 \cdots \wedge d\bar{z}_n$$

$$= (\text{power of } i) f(z_1, \ldots, z_n, \bar{z}_1, \ldots \bar{z}_n) dx_1 \wedge dy_1 \cdots \wedge dx_n \wedge dy_n$$

POINCARÉ DUAL FORM: COMPLEX CASE

If N is a k-dimensional complex submanifold of an n dimensional compact Kähler manifold M, then

For $\omega \in A^{p,q}(M)$, with $p+q=2k$ and $d\omega=0$,

$\int_N \omega = 0$ unless $(p,q)=(k,k)$

We can use this to arrange that

$\eta_N \in A^{n-k,n-k}(M)$

HODGE CLASSES

We now let X be a smooth projective algebraic variety and Z an algebraic subvariety of X

The **codimension** of Z is $\dim(X) - \dim(Z)$

Let Z have codimension p

Now $H^{2p}_{DR}(X, \mathbf{C}) \cong H^{2p}_{\text{sing}}(X, \mathbf{C})$

In topology, we also have the **integral cohomology**

$H^{2p}_{\text{sing}}(X, \mathbf{Z})$

and the coefficient map

$j \colon H^{2p}_{\text{sing}}(X, \mathbf{Z}) \to H^{2p}_{\text{sing}}(X, \mathbf{C})$

Because Z represents a topological $2\dim(Z)$-chain, from topology we get that

$[\eta_Z] \in H^{p,p}(X) \cap j(H^{2p}_{\text{sing}}(X, \mathbf{Z})) \subset H^{2p}_{\text{sing}}(X, \mathbf{C})$

We define

$\text{Hdg}^p(X) = H^{p,p}(X) \cap j(H^{2p}_{\text{sing}}(X, \mathbf{Z})),$

the **Hodge classes** of X in codimension p

Theorem: The Poincaré dual class of an algebraic subvariety is a Hodge class

ALGEBRAIC CYCLES

By analogy with what is done in topology, we define the **codimension p algebraic cycles** $Z^p(X) =$

$\{\sum_i n_i Z_i \mid n_i \in \mathbf{Z}, Z_i$ a codim p alg subvariety for all $i, n_i \in \mathbf{Z}$ for all $i\}$

We think of these as formal linear combinations

There is a map

$\eta\colon Z^p(X) \to \mathrm{Hdg}^p(X)$ given by

$\sum_i n_i Z_i \mapsto \sum_i n_i [\eta_{Z_i}]$ (**Cycle class map**)

Originally, Hodge conjectured that this map is surjective. Atiyah and Hirzebruch found a counterexample, but this involved a torsion phenomenon:

The Hodge Conjecture: For any Hodge class on a smooth projective algebraic variety, some non-zero integral multiple of it is the Poincaré dual class for some algebraic cycle

Another way to put this is:

$Z^p(X, \mathbf{Q}) = \{\sum_i r_i Z_i \mid r_i \in \mathbf{Q}\}$

$\mathrm{Hdg}(X, \mathbf{Q}) = H^{p,p}(X) \cap j(H^{2p}_{\mathrm{sing}}(X, \mathbf{Q}))$

$\eta_{\mathbf{Q}}\colon Z^p(X, \mathbf{Q}) \to \mathrm{Hdg}^p(X, \mathbf{Q})$

defined analogously

The Hodge Conjecture: $\eta_{\mathbf{Q}}$ is surjective

THE CASE OF SURFACES OF DEGREE d

For X a smooth surface of degree d in $\mathbf{CP}^3$, and C an algebraic curve on X,

$[\eta_C] \in \mathrm{Hdg}^1(X)$

Now, there is a distinguished Hodge class: if H is a hyperplane, then

$[\eta_{X\cap H}]$ is called the **hyperplane class**

Using some topology, we can show that the hyperplane class is not a non-trivial integral multiple of another integral cohomology class

For a complete intersection curve $X \cap X'$, where X' has degree d',

$[\eta_{X\cap X'}] = d'[\eta_{X\cap H}]$

With some work using standard algebraic geometry plus some topology,

An algebraic curve C on X is a complete intersection on X if and only if $[\eta_C]$ is an integral multiple of $[\eta_{X\cap H}]$

This successfully rephrases the Noether-Lefschetz problem as:

Hodge-theoretic Noether-Lefschetz: Show that for almost all surfaces of degree $d \geq 4$,

$\mathrm{Hdg}^1(X) = \mathbf{Z}[\eta_{X\cap H}]$

FAMILIES OF COMPLEX MANIFOLDS

A **holomorphic family of complex manifolds**

$\pi: \mathcal{X} \to S$

is a complex manifold such that S is a complex manifold, π is analytic and $d\pi$ is surjective at all points. It follows that

$X_s = \pi^{-1}(s)$

is a complex manifold for all s of dimension $\dim(\mathcal{X}) - \dim(S)$

S is called the **parameter space**

If S is contractible, then all of the X_s are diffeomorphic, and there is a natural homotopy class (or even isotopy class) of diffeomorphisms between X_{s_1}, X_{s_2} for any $s_1, s_2 \in S$

In particular, $H^k_{\text{sing}}(X_{s_1}, \mathbf{Z})$, $H^k_{\text{sing}}(X_{s_2}, \mathbf{Z})$ have a natural isomorphism between them

When $\pi_1(S) \neq 0$, there is a group homomorphism

$\rho: \pi_1(S, s) \to \mathrm{Aut}(H^k_{\text{sing}}(X_s, \mathbf{Z}))$

called the **monodromy representation**

Analyzing the monodromy is the key concept in Lefschetz's proof of Noether-Lefschetz.

FAMILIES OF HYPERSURFACES OF DEGREE d

If we look at

$F + sG$,

where F, G are hypersurfaces of degree d and F is smooth, then one can show

$X_s = \{F + sG = 0\}$ is smooth for $|s| < \epsilon$ for some small $\epsilon > 0$

Similarly for

$F + \sum_I s_I z^I$, where

$I = (i_1, i_2, i_3, i_4)$ with all $i_j \in \mathbf{Z}^{\geq 0}$ and $i_1 + \cdots + i_4 = d$

$z^I = z_1^{i_1} z_2^{i_2} z_3^{i_3} z_4^{i_4}$

which is smooth for small values of the s_I

VARIATION OF HODGE STRUCTURE

Given a smooth family $\mathcal{X}$ of Kähler manifolds with S contractible, we can think of

$H^k_{\text{sing}}(X_s, \mathbf{C})$ as a fixed vector space V

and the Hodge decomposition as varying with s

Griffiths discovered:

$H^{p,q}(X_s)$, as a subspace of $H^k_{DR}(X_s, \mathbf{C})$, does not vary analytically

but the **Hodge filtration**

$F^p H^k_{DR}(X_s, \mathbf{C}) = \sum_{p' \geq p} H^{p', k-p'}(X_s)$ does vary analytically as a subspace of $H^k_{DR}(X_s, \mathbf{C})$

For example,

$F^2 H^2(X_s, \mathbf{C}) = H^{2,0}(X_s)$

$F^1 H^2(X_s, \mathbf{C}) = H^{2,0}(X_s) \oplus H^{1,1}(X_s)$

$F^0 H^2(X_s, \mathbf{C}) = H^{2,0}(X_s) \oplus H^{1,1}(X_s) \oplus H^{0,2}(X_s) = H^2(X_s, \mathbf{C})$

Note $F^k \subseteq F^{k-1} \subseteq \cdots \subseteq F^0 = H^k_{DR}(X, \mathbf{C})$,

i.e. the Hodge filtration is a decreasing filtration

$F^p / F^{p+1} \cong H^{p,k-p}(X_s)$

DERIVATIVE OF A VARIABLE SUBSPACE OF A FIXED VECTOR SPACE

W_s an analytically varying subspace of a fixed vector space V

Choose $e_1(s), \ldots, e_m(s)$ an analytically varying basis for W_s

$e_j \mapsto \pi_{V/W_s}(de_j/ds)$ gives a well-defined linear map

$d/ds\colon W_s \to V/W_s$

If we have many variables, we get

$T_s S \to \mathrm{Hom}_{\mathbf{C}}(W_s, V/W_s)$

$\partial/\partial s_i \mapsto (e_j \mapsto \pi_{V/W_s}(\partial e_j/\partial s_i))$

A more elegant way to think of this is

$T_s S \otimes W_s \to V/W_s$

$\partial/\partial s_i \otimes e_j \mapsto \pi_{V/W_s}(\partial e_j/\partial s_i)$

GRIFFITHS TRANSVERSALITY
aka
INFINITESIMAL PERIOD RELATION

If we look at

$d/ds(e_1(s) \wedge e_2(s) \wedge e_k(s))$

$= (de_1/ds) \wedge e_2 \wedge \cdots \wedge e_k + \cdots + e_1 \wedge e_2 \cdots \wedge (de_k/ds)$

we see that the derivative of something with p dz's and q $d\bar{z}$'s has at least $p-1$ dz's

The upshot is that for all p,

$d/ds \colon F^p \to V/F^p$

actually lands in

F^{p-1}/F^p

This discovery is known as **Griffiths transversality** or the **infinitesimal period relation**

Note that we get maps

$T_sS \otimes F^p/F^{p+1} \to F^{p-1}/F^p$, and thus

$T_sS \otimes H^{p,k-p}(X_s) \to H^{p-1,k-p+1}(X_s)$

For surfaces, this gives us a map

$T_sS \otimes H^{2,0}(X_s) \to H^{1,1}(X_s)$

A COMMENT ABOUT DERIVATIVES

A basic principle about differentiable functions is:

If $df_{s_0} \neq 0$, then we cannot have $f \equiv 0$ on S

For $\gamma \in Hdg^1(X_{s_0})$, we may write for nearby s

$\gamma = \gamma^{2,0}(s) + \gamma^{1,1}(s) + \gamma^{0,2}(s)$

in $H^2_{DR}(X_s)$

Now

$\gamma^{0,2}(s_0) = 0$, but if $d\gamma^{0,2}_{s_0} \neq 0$, then γ cannot be of type $(1,1)$ on all of S, and hence

γ can only be a Hodge class on a lower-dimensional subset of S

Because $\gamma^{0,2}(s)$ is an analytic section of the analytic bundle F^0/F^1, its zero locus

$\{s \mid \gamma \in Hdg^1(X_s)\}$

is locally defined by analytic functions, i.e. the zero locus is locally an analytic subvariety of S

Unless $\gamma^{0,2}(s) \equiv 0$ for all s, this zero locus will be a lower-dimensional analytic subvariety of S

DERIVATIVE OF A HODGE CLASS

In a smooth family $\mathcal{X} = \{X_{s_0}\}$, if $\gamma \in H^{2p}_{DR}(X_{s_0})$ is a Hodge class, we may take γ to be part of a basis for F^p

As we vary in the family, γ continues to be an integral class, but it may not continue to be in $H^{1,1}$

The condition that it does remain a Hodge class is that under

$\partial/\partial s_j\colon H^{p,p}(X_s) \to H^{p-1,p+1}(X_s)$,

$\gamma \mapsto 0$ for all s and all j, or more elegantly

$T_sS \otimes H^{p,p}(X_s) \to H^{p-1,p+1}(X_s)$

has γ in the right kernel

Using some dualities, this turns out to be equivalent to saying

γ is orthogonal to the image of $T_sS \otimes H^{p+1,p-1} \to H^{p,p}$

for all s

(A map $A \otimes B \to C$ gives a map $A \otimes C^\vee \to B^\vee$

The right kernal of $A \otimes B \to C$ is dual to the cokernel of $A \otimes C^\vee \to B^\vee$

$A \otimes B \to C$ has right kernel zero $\Leftrightarrow A \otimes C^\vee \to B^\vee$ is surjective)

WHEN THE SET OF s FOR WHICH X_s HAS A HODGE CLASS IS A LOWER-DIMENSIONAL SUBVARIETY

A consequence of the foregoing is:

THEOREM: If at some point,

$T_sS \otimes H^{p+1,p-1}(X_s) \to H^{p,p}(X_s)$ **is surjective,**

then the set of s for which X_s has a Hodge class is a lower-dimensional analytic subvariety of S

It is actually the union of a countable number of proper subvarieties of S

For hypersurfaces or for projective varieties in general, since $[H \cap X_s]^p$ is always a Hodge class, we modify this to looking at $([H \cap X_s]^p)^\perp \subseteq H^{p,p}(X_s)$

THEOREM: If at some point,

$T_sS \otimes H^{p+1,p-1}(X_s) \to ([H \cap X_s]^p)^\perp \subseteq H^{p,p}(X_s)$ **is surjective,**

then the set of s for which X_s has a Hodge class other than rational multiples of $[H \cap X_s]^p$ is a lower dimensional analytic subvariety of S

The set of s where there are Hodge classes other than rational multiples of $[H \cap X_s]^p$ is the **Noether-Lefschetz locus**

We have reduced the Noether-Lefschetz Theorem to asking whether $T_sS \otimes H^{2,0}(X_s) \to ([H \cap X_s])^\perp$ is surjective

HODGE GROUPS OF SURFACES OF DEGREE d

$F = F(z_1, z_2, z_3, z_4)$ a homogeneous polynomial of degree d

$J(F) =$ ideal generated by $\partial F/\partial z_1, \ldots, \partial F/\partial z_4$, the **Jacobi ideal**

$X = \{F = 0\}$ is smooth $\Leftrightarrow J(F)$ is base-point free

Base point free for an ideal means that there is no z other than $(0, 0, 0, 0)$ for which all polynomials in the ideal vanish

Let $V^m = \{$homogeneous polynomials of degree $m\}$

For I a homogeneous ideal,

I_m denotes $\{$homogeneous degree m part of $I\}$

$H^{2,0} \cong V^{d-4}$

$[X \cap H]^\perp \subset H^{1,1} \cong V^{2d-4}/J(F)_{2d-4}$

We denote $[X \cap H]^\perp = H^{1,1}_{\mathrm{pr}}(X)$

Note that for $d < 4$, all integral classes in $H^2(X)$ are Hodge classes

This explains the role of $d \geq 4$ in the Noether-Lefschetz theorem

TANGENT TO THE PARAMETER SPACE

$S = \{\text{hypersurfaces of degree } d\}/\text{projective equivalence}$

Here, **projective equivalence** is the action of $GL(4, \mathbf{C})$ on $\mathbf{CP}^3$, taking $X \mapsto gX$

Tangent to the action of $GL(4, \mathbf{C})$ on $\mathbf{CP}^3$ are the global vector fields on $\mathbf{CP}^3$, whose action on X gives the tangent space to projective equivalence

These vector fields are spanned by $z_i\partial/\partial z_j$ for $1 \leq i, j \leq 4$

The first order action of $z_i\partial/\partial z_j$ on F is $z_i\partial F/\partial z_j$

$T_F(V^d/\text{projective equivalence}) \cong V^d/J(F)_d$

DERIVATIVE OF THE HODGE GROUPS FOR SURFACES OF DEGREE d

THEOREM (Carlson-Griffiths): The map

$T_F S \otimes H^{2,0}(X) \to H^{1,1}_{\mathrm{pr}}(X)$, i,e, of

$V^d / J(F)_d \otimes V^{d-4} \to V^{2d-4}/J(F)_{2d-4}$

is multiplication, i.e.

$G \otimes P \mapsto GP$

Easy result: Multiplication $V^a \otimes V^b \to V^{a+b}$ is surjective when $a, b \geq 0$

Proof: It is enough to see that every monomial of degree $a+b$ is the product of monomials of degrees a and b

THEOREM (Infinitesimal Noether-Lefschetz Thm)

$T_F S \otimes H^{2,0}(X) \to H^{1,1}_{\mathrm{pr}}(X)$ **is surjective when** $d \geq 4$

COROLLARY (Noether Lefschetz Theorem)

This argument is due to Carlson-G-Griffiths-Harris

EXPLICIT NOETHER-LEFSCHETZ THEOREM

There is a result from commutative algebra that is enough to show that every component of the Noether-Lefschetz locus has codimension $\geq d-3$

$W \subseteq V^d$ a base-point free linear subspace

$c = \dim(V^d/W)$

For $b \geq c$,

$W \otimes V^b \to V^{d+b}$ is surjective

In the geometric case, we have that if S is the parameter space of an irreducible component of the Noether-Lefschetz locus and

$T_F S = W$, then $J(F)_d \subseteq W$, which for F smooth forces W to be base-point free

The algebraic result is part of a general Koszul vanishing theorem I proved. I eventually found another proof using results of Macaulay and Gotzmann

LECTURE 3: Hodge Structures and Mumford-Tate Domains

STRUCTURE OF ROTATIONS IN $\mathbf{R}^2$ AND $\mathbf{R}^3$

$SO(n) = \{A \in GL(n, \mathbf{R}) \mid A^t A = I, \det(A) = 1\}$

$= \{A \in GL(n, \mathbf{R}) \text{ preserving angles, lengths, and orientation}\}$

$= \{A \mid \text{columns have length } 1, \text{pairwise perpendicular}\}$

$$SO(2) = \{R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}\}$$

$SO(3) = \{\text{rotations about some axis}\}$

Proof: $A \in SO(3)$ has an eigenvector v, $Av = \lambda v$, $|\lambda| = 1$

One root of characteristic polynomial, a cubic, is real. That λ is ± 1

Complex eigenvectors come in conjugate pairs, and then $\lambda\bar{\lambda} > 0$. So for one real eigenvalue λ and a conjugate $\mu, \bar{\mu}$, $\lambda\mu\bar{\mu} = 1$, so $\lambda > 0$, so $\lambda = 1$

For three real eigenvalues, $\lambda_1\lambda_2\lambda_3 = 1$, all ± 1, so one of these must be $+1$

If v has eigenvalue $+1$, then if $V = v^\perp$, $A|_V \in SO(2)$

There is thus a choice of properly oriented orthonormal basis so that A has the matrix

$$A = \begin{pmatrix} R_\theta & 0 \\ 0 & 1 \end{pmatrix}$$

Another way to say this is that A is conjugate in $SO(3)$ to a matrix of this form

STRUCTURE OF ROTATIONS IN $\mathbf{R}^n$

A similar argument shows that $A \in SO(n)$ has a matrix in terms of some properly oriented orthonormal basis

$$A = \begin{pmatrix} R_{\theta_1} & 0 & \dots & 0 \\ 0 & R_{\theta_2} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & R_{\theta_k} \end{pmatrix} \text{ if } n = 2k \text{ and}$$

$$A = \begin{pmatrix} R_{\theta_1} & 0 & \dots & 0 & 0 \\ 0 & R_{\theta_2} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & \dots & R_{\theta_k} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \text{ if } n = 2k+1$$

Alternatively we can say every $A \in SO(n)$ is conjugate in $SO(n)$ to a matrix of this form

The set of matrices of either of these forms in a given basis is isomorphic to

$T = S^1 \times S^1 \times \cdot \times S^1 = (S^1)^k$, a k-dimensional torus

Every element of $SO(n)$ is conjugate to an element of T

LIE GROUPS AND THEIR LIE ALGEBRAS

A **Lie group** G is a group that is also a smooth manifold, such that multiplication and inversion are $\mathcal{C}^\infty$ maps

Left multiplication $L_g \colon G \to G$ is the map

$h \mapsto gh$

So $dL_g \colon T_eG \cong T_gG$

We may use this to extend $X \in T_eG$ to a smooth **left invariant vector field** on G by $X(g) = (dL_g)_e(X)$ for all $g \in G$

The **Lie algebra of** G is

$\mathbf{g} = T_eG = \{\text{Left invariant vector fields on } G\}$

On a manifold M, vector fields X, Y on M, there is a vector field $[X, Y]$ defined by, for functions f on M,

$[X, Y]f = X(Yf) - Y(Xf)$, the **Lie bracket** of X and Y

It is a measure of the failure of differentiation along X and Y to commute

If X,Y are left-invariant, then so is $[X, Y]$, so

$X, Y \in \mathbf{g}$ gives an element $[X, Y] \in \mathbf{g}$

This satisfies the **Jacobi identity**

$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$

ACTIONS OF S^1

Consider a continuous group homomorphism

$\phi\colon S^1 \to \mathrm{Aut}(V, \mathbf{R})$

If $<,>$ is a positive definite inner product on V, then we can create an S^1-invariant positive definite inner product by

$< v, w >_{\mathrm{inv}} = \int_0^{2\pi} < \phi(\theta)v, \phi(\theta)w > d\theta$

Using this inner product, $\phi\colon S^1 \to O(n)$, where $n = \dim(V)$

Now $\det(\phi(\theta)) = \pm 1$, is continuous in θ, and is 1 at 0 because $\theta(0) = I$. So

$\phi\colon S^1 \to SO(n)$

Now the $\phi(\theta)$ mutually commute since S^1 is commutative, so since each of them can be diagonalized, they can be simultaneously diagonalized. Hence after an orientation preserving orthonormal change of basis,

$\phi\colon S^1 \to T \subseteq SO(n)$

WEIGHTS OF S^1 ACTIONS

Group homomorphisms $S^1 \to (S^1)^k$ are of the form

$\theta \mapsto (m_1\theta, m_2\theta, \ldots, m_k\theta)$, where $m_i \in \mathbf{Z}$ for all i

More abstractly, if $T = \mathbf{t}/\Lambda$ with $\mathbf{t} \cong \mathbf{R}^k$ the Lie algebra of T as a group and $\Lambda \cong \mathbf{Z}^k$ a lattice, then

$\vec{m} = (m_1, \ldots, m_k) \in \Lambda$ is called the **weight** of ϕ, and we can write $\phi_{\vec{m}}$ for ϕ

EIGENSPACES OF S^1 ACTIONS

The eigenvalues of R_θ are $e^{i\theta}$, $e^{-i\theta}$

The eigenvalues of $\phi_{\vec{m}}$ are $e^{\pm i m_1 \theta}, \ldots, e^{\pm i m_k \theta}$

We can write the complexification of V as

$V_{\mathbf{C}} = \bigoplus_j V^j$, where the direct sum is orthogonal and

$\phi(\theta)$ is $e^{ji\theta} I$ on V^j, i.e.

V^j is the $e^{ji\theta}$ eigenspace of $\phi(\theta)$ for all θ

If $\phi = \phi_{\vec{m}}$, then V^j is non-zero only when $j = \pm m_q$ for some q

INTERSECTION PAIRING

For a smooth projective variety X of dimension n we have that

$H^{2n}_{DR}(X) \cong \mathbf{R}$

under the map

$\omega \mapsto \int_X \omega$

We thus get a map

$H^n_{DR}(X) \times H^n_{DR}(X) \to H^{2n}_{DR}(X) \cong \mathbf{R}$

$(\omega_1, \omega_2) \mapsto \int_X \omega_1 \wedge \omega_2$

this is symmetric in ω_1, ω_2 for n even and alternating for n odd

It comes from the cup product map

$Q\colon H^n_{\text{sing}}(X, \mathbf{Z}) \times H^n_{\text{sing}}(X, \mathbf{Z}) \to H^{2n}_{\text{sing}}(X, \mathbf{Z}) \cong \mathbf{Z}$

We call Q the **intersection pairing**

Comment: The key case in Hodge theory is looking at $H^n(X)$ for X of complex dimension n. We can reduce to this case because of the **Lefschetz hyperplane theorem**

POLARIZED HODGE STRUCTURES OF WEIGHT n

Set-up: V a finite dimensional real vector space, $\Lambda \subset V$ a lattice,

$\Lambda \cong \mathbf{Z}^m$,

$V = \Lambda \otimes_{\mathbf{Z}} \mathbf{R}$, so $V \cong \mathbf{R}^m$

$Q\colon \Lambda \times \Lambda \to \mathbf{Z}$ a non-degenerate $\mathbf{Z}$-bilinear map

$Q(y, x) = (-1)^n Q(x, y)$ for all $x, y \in \Lambda$

$V_{\mathbf{C}} \cong \bigoplus_{p+q=n} V^{p,q}$, with $V^{q,p} = \bar{V}^{p,q}$

$Q(V^{p,q}, V^{p',q'}) = 0$ unless $p' = q, q' = p$

Note $i^{p-q}Q = i^{n-2q}Q$ is Hermitian. We ask that

Positivity condition: $i^{p-q}Q$ is positive definite on $V^{p,q}$

If so, we say that (V, Λ, Q) is a **polarized Hodge structure of weight** n

Comment: For X a smooth projective variety of dimension n, it is almost true that $(H^n_{DR}(X), H^n_{\text{sing}}(X, \mathbf{Z}), \text{intersection pairing})$ is a polarized Hodge structure. To make this work, we need to restrict to the **primitive cohomology**. For $n = 2$, the primitive cohomology is $[H \cap X]^\perp \subset H^2_{DR}(X)$. The positivity condition on the intersection form on the primitive cohomology is a piece of the **Hodge index theorem**, and incorporates the information in the Lefschetz hyperplane theorem

WHERE POSITIVITY COMES FROM

$dz \wedge d\bar{z} = (dx + idy) \wedge (dx - idy) = -2idx \wedge dy$

If $\omega = f(z)dz$ locally, then

$\omega \wedge \bar{\omega} = |f(z)|^2 dz \wedge d\bar{z} = -2i|f(z)|^2 dx \wedge dy$

So if $\omega \in H^{1,0}(X)$ for a Riemann surface X,

$i^{p-q} = i^{1-0} = i$, and

$i \int_X \omega \wedge \bar{\omega}$ gets local contribution $\int 2|f(z)|^2 dx \wedge dy > 0$ from this coordinate patch

Similarly, if $\omega \in H^{2,0}(X)$ for X a surface,

$\omega = f(z_1, z_2)dz_1 \wedge dz_2$ locally, then

$\omega \wedge \bar{\omega} = |f|^2 dz_1 \wedge dz_2 \wedge d\bar{z}_1 \wedge d\bar{z}_2$

$= -|f|^2 dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2$

$= -(-2i)^2 |f|^2 dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2$

$= -4|f|^2 dx_1 \wedge dx_2 \wedge dy_1 \wedge dy_2$

So $i^{2-0} \int_X \omega \wedge \bar{\omega}$ gets local contribution

$\int 4|f|^2 dx_1 \wedge dx_2 \wedge dy_1 \wedge dy_2 > 0$

Comment: We are choosing our orientation so $dx_1 \wedge dx_2 \wedge dy_1 \wedge dy_2$ is properly oriented–this is unchanged in sign if we switch the order of z_1 and z_2, as the switches of dx_1, dx_2 and dy_1, dy_2 cancel out

The case of (1,1) forms is more complicated, and this is where primitivity shows up

POLARIZED HS: ELEGANT VERSION

With V, Λ, Q as before, let

$\phi: S^1 \to \mathrm{Aut}(V, \mathbf{R})$ be a continuous group homomorphism such that

Q is $\phi(\theta)$-invariant for all θ

Let $V_{\mathbf{C}} = \bigoplus V^j$ be the eigenspace decomposition for ϕ

Assume these are non-zero only for j with $j \equiv n \pmod 2$

Define $V^{(n+j)/2,(n-j)/2} = V^j$ for all j

Put another way, $V^{p,q} = V^{p-q}$ for $p + q = n$

The condition to have a polarized Hodge structure is then:

$i^j Q$ is positive definite on V^j

Thus:

A polarized Hodge structure of weight n is a continuous group homomorphism $\phi: S^1 \to \mathrm{Aut}(V, \mathbf{R})$ such that Q is ϕ-invariant, non-zero eigenspaces V^j have the same parity as n and $i^j Q$ is positive definite on V^j for all j. We set $V^{p,q} = V^{p-q}$ for $p + q = n$

Comment: Not surprisingly, this elegant way of doing things goes back to Deligne

GROUP REPRESENTATIONS

Given a Lie group G, and a finite dimensional real vector space V, a continuous group homomorphism

$\rho\colon G \to \mathrm{Aut}(V, \mathbf{R})$ is called a **representation of** G over $\mathbf{R}$

If Q is a non-degenerate symmetric or alternating bilinear form on V,

$\mathrm{Aut}_{\mathbf{R}}(V, Q)$ will denote the real linear automorphisms of V preserving Q

ADJOINT REPRESENTATION

Given a Lie group G and $g \in G$, conjugation gives a map

$c_g\colon G \to G$

$h \mapsto ghg^{-1}$

The derivative of c_g at the identity is a map

$dc_g\colon T_eG \to T_eG$

Identifying $T_eG = \mathbf{g}$, we can rewrite this as

$dc_g\colon \mathbf{g} \to \mathbf{g}$, and we denote $\mathrm{Ad}_g = dc_g$, giving a map

$\mathrm{Ad}\colon G \to \mathrm{Aut}_{\mathbf{R}}(\mathbf{g})$, the **adjoint representation**

The derivative of this map at the identiy in turn gives a map

$d\mathrm{Ad}_e\colon T_eG \to \mathrm{End}_{\mathbf{R}}(\mathbf{g})$

Identifying $T_eG = \mathbf{g}$, we get a map

$d\mathrm{Ad}_e\colon \mathbf{g} \to \mathrm{End}_{\mathbf{R}}(\mathbf{g})$

We call this map

$\mathrm{ad}\colon \mathbf{g} \to \mathrm{End}_{\mathbf{R}}(\mathbf{g})$

$X \mapsto \mathrm{ad}_X$

Key formula: $\mathrm{ad}_X(Y) = [X, Y]$, i.e.

$\mathrm{ad}_X = [X, \cdot]$

REPRESENTATIONS OF LIE ALGEBRAS

For a vector space V, $\mathrm{End}_{\mathbf{R}}(V)$ can be made into a Lie algebra by setting

$[A, B] = AB - BA$

$\mathrm{End}_{\mathbf{R}}(V)$ is the Lie algebra of $\mathrm{Aut}_{\mathbf{R}}(V)$

A linear map

$r\colon \mathbf{g} \to \mathrm{End}_{\mathbf{R}}(V)$

is called a **Lie algebra representation** of $\mathbf{g}$ if

$r([X, Y]) = [r(X), r(Y)]$ for all $X, Y \in \mathrm{g}$

Jacobi relation $\Leftrightarrow$ $\mathrm{ad}_{[X,Y]} = [\mathrm{ad}_X, \mathrm{ad}_Y]$ for all $X, Y \in \mathbf{g}$

This is equivalent to saying that

$\mathrm{ad}\colon \mathbf{g} \to \mathrm{End}_{\mathbf{R}}(\mathbf{g})$ is a representation of Lie algebras

CARTAN-KILLING FORM

We define a symmetric bilinear form on $\mathbf{g}$

$Q(X, Y) = \mathrm{Tr}(\mathrm{ad}_X \mathrm{ad}_Y)$, the **Cartan-Killing form**

This has the nice invariance property that

$\mathrm{Ad}\colon G \to \mathrm{Aut}_{\mathbf{R}}(\mathbf{g}, Q)$, i.e.

Q is invariant under the adjoint representation

A Lie group is called **semisimple** if the Cartan-Killing form is non-degenerate

Equivalently, this says that $\mathrm{ad}\colon \mathbf{g} \to \mathrm{End}_{\mathbf{R}}(\mathbf{g})$ is injective

A semisimple Lie algebra is **simple** if $\mathbf{g}$ is not a non-trivial direct sum of two semisimple Lie algebras

In the simple case, the only invariant symmetric bilinear forms are multiples of the Cartan-Killing form

Comment: $\mathrm{Tr}(AA^t) > 0$ for all non-zero real matrices A. So if ad_X is represented by a symmetric matrix, $Q(X, X) > 0$ and if by an antisymmetric matrix, $Q(X, X) < 0$

HODGE STRUCTURES ARISING FROM REPRESENTATIONS

Given a representation

$\rho: G \to \mathrm{Aut}_{\mathbf{R}}(V, Q)$, and a continuous group homomorphism

$\psi: S^1 \to G$,

we may ask when $\phi = \rho \circ \psi$ gives a polarized Hodge structure on V

A REMARKABLE TRICK

If $\phi = \rho \circ \psi$ gives a polarized Hodge structure on V, by using $\phi \otimes \phi^\vee$, we get a polarized Hodge structure $\tilde{\phi}$ on $\mathrm{End}_{\mathbf{R}}(V) \cong V^\vee \otimes_{\mathbf{R}} V$

Now $d\rho\colon \mathbf{g} \to \mathrm{End}_{\mathbf{R}}(V)$, which is injective if the representation is **faithful**, i.e. ρ is injective

Further, in case $\mathbf{g}$ is simple, $d\rho$ pulls back the Q on V to a non-zero multiple of the Cartan-Killing form, by uniqueness of the invariant forms on $\mathbf{g}$

One checks $d\rho \circ \mathrm{Ad} = \tilde{\phi}$, and thus $\mathbf{g}$ is invariant under $\tilde{\phi}$, and thus is a direct sum of eigenspaces of $\tilde{\phi}$

Since Ad preserves the Cartan-Killing form, and since the signs (after possibly multiplying the whole thing by -1) are correct to have a polarized Hodge structure, we get that $\mathrm{Ad} \circ \psi$ gives a polarized Hodge structure on $\mathbf{g}$

Theorem: For a simple g, if any faithful representation of G has a polarized Hodge structure, it induces a polarized Hodge structure on g for the adjoint representation and the Cartan-Killing form

MAXIMAL COMPACT SUBGROUPS

For a semisimple Lie group G, a **maximal compact subgroup** K is a maximal connected compact subgroup. These are all conjugate.

Picking a maximal compact subgroup K, its Lie algebra $\mathbf{k} \subseteq \mathbf{g}$

Denote the orthogonal complement under the Cartan-Killing form

$\mathbf{p} = \mathbf{k}^{\perp}$, so

$\mathbf{g} = \mathbf{k} \oplus \mathbf{p}$

It turns out that the Cartan-Killing form satisfies:

$Q < 0$ on $\mathbf{k}$,

$Q > 0$ on $\mathbf{p}$, and

$Q(\mathbf{k}, \mathbf{p}) = 0$ by construction

We may take

$\mathbf{k} = \{X \in \mathbf{g} \mid \mathrm{ad}_X \text{ is antisymmetric}\}$

$\mathbf{p} = \{X \in \mathbf{g} \mid \mathrm{ad}_X \text{ is symmetric}\}$

POLARIZED HODGE STRUCTURES ON $\mathbf{g}$

When we unwind the conditions, for $\phi\colon S^1 \to G$ to give a polarized Hodge structure for the adjoint representation, we have (for $\mathbf{g}$ simple) that there is an invariant symmetric from–the Cartan-Killing form–and no invariant anti-symmetric form. So we must have an even weight Hodge structure, and

$Q = \pm$Cartan Killing form

Because $d\phi_0(\partial/\partial\theta) \in \mathbf{g}$ is in the image of a circle, it must be compact, but

$d\phi_0(\partial/\partial\theta) \in \mathbf{g}^0$ by commutativity of S^1, so in order for Q to be positive-definite on $\mathbf{g}^0$, we must take

$Q = -$Cartan Killing form

Because the weight of the Hodge structure is even, the eigenvalues of $\mathrm{Ad} \circ \phi$ must all be even, and the condition to be polarized is

$\mathbf{k} = \bigoplus_{j \equiv 0 (\mathrm{mod}\ 4)} \mathbf{g}^j$, and

$\mathbf{p} = \bigoplus_{j \equiv 2 (\mathrm{mod}\ 4)} \mathbf{g}^j$,

i.e. the eigenvalues are all even, and the eigenvectors are contained in $\mathbf{k}$ for eigenvalues divisible by 4 and in $\mathbf{p}$ otherwise

CARTAN SUBALGEBRAS AND RANK

$\mathbf{g}$ a semisimple real Lie algebra

A **Cartan subalgebra** $\mathbf{h} \subset \mathbf{g}$ is a maximal abelian subalgebra whose elements X all have ad_X diagonalizable over $\mathbf{C}$

All Cartan subalgebras of $\mathbf{g}$ have the same dimension r, called the **rank** of $\mathbf{g}$ (or of G)

A Cartan subgroup of G is a connected closed Lie subgroup whose Lie algebra is a Cartan subalgebra

Every commuting set of diagonalizable elements of $\mathbf{g}$ is contained in some Cartan subalgebra

In particular, if $\phi\colon S^1 \to G$ is a continuous group homomorphism, $\Phi = d\phi_0(\partial/\partial\theta) \in \mathbf{g}$ belongs to a Cartan subalgebra $\mathbf{h}$

CONDITION FOR $\mathbf{g}$ TO HAVE A POLARIZED HODGE STRUCTURE

Let $\mathbf{g}$, $\Phi \in \mathbf{h}$ as in the previous slide

If $\mathrm{Ad} \circ \phi$ gives a polarized Hodge structure on $\mathbf{g}$, then by the earlier argument, we have

$\mathbf{g}^0 \subseteq \mathbf{k}$

$[\Phi, X] = 0$ for all $X \in \mathbf{h}$, and exponentiating,

$\mathbf{h} \subseteq \mathbf{g}^0 \subseteq \mathbf{k}$

So: $\mathbf{g}$ has a Cartan subalgebra contained in $\mathbf{k}$

Exponentiating,

G has a Cartan subgroup that is a compact real torus T,

i.e. $\dim(T) = \mathrm{rank}(G)$, and

$\phi \colon S^1 \to T \subset G$

This condition was first noticed by Carlos Simpson

This gives one direction of the following result:

THEOREM: The adjoint representation of a semisimple Lie group G can be given a polarized Hodge structure $\Leftrightarrow$ G contains a compact real torus which is a Cartan subgroup

As mentioned, if any representation of G can be given a polarized Hodge structure factoring through G, then the adjoint representation of G can

MUMFORD-TATE GROUPS

In our new setting, Hodge classes if $n = 2p$ are

$Hdg^p = V^{p,p} \cap \Lambda$

If $(V_1, \Lambda_1, Q_1, \phi_1)$, $(V_2, \Lambda_2, Q_2, \phi_2)$ are polarized Hodge structures of weights n_1, n_2, then

$(V_1 \otimes_{\mathbf{R}} V_2, \Lambda_1 \otimes_{\mathbf{Z}} \Lambda_2, Q_1 \otimes_{\mathbf{Z}} Q_2, \phi_1 \otimes_{\mathbf{R}} \phi_2)$ inherits a polarized Hodge structure of weight $n_1 + n_2$

For the dual $V^\vee$,

$(V^\vee, \Lambda^\vee, Q^\vee, \phi^\vee)$ inherits a polarized Hodge structure

The **Mumford-Tate group** of a polarized Hodge structure is

$G = \{g \in \mathrm{Aut}_{\mathbf{R}}(V, Q) \mid g$ fixes all Hodge classes in

$\bigotimes^a V \otimes \bigotimes^b V^\vee$ for all $a, b\}$

The equations defining G inside $GL(V, \mathbf{R})$ are polynomials with coefficients in $\mathbf{Z}$, i.e. it is a **linear algebraic group** defined over $\mathbf{Q}$

Comment: When G is a linear algebraic group over $\mathbf{Q}$, its Lie algebra $\mathbf{g}$ naturally gets the structure of a vector space over $\mathbf{Q}$. The lattice Λ that we need in order to have a Hodge structure may be chosen compatible with this rational structure, and so the Cartan-Killing form is integer-valued on $\Lambda \times \Lambda$

WHICH CONNECTED REAL SEMISIMPLE LIE GROUPS CAN BE MUMFORD-TATE GROUPS

THEOREM: (G-Griffiths-Kerr) A connected semisimple real Lie group can be a Mumford-Tate group if and only if G contains a compact real torus which is a Cartan subgroup

One can use classification of simple Lie algebras to actually make a list

Comment: There is a more refined question where we ask which semisimple linear algebraic groups G defined over $\mathbf{Q}$ can be Mumford-Tate groups. Our result is only about the induced structure on G as a semisimple real Lie group.

THE PLOT THICKENS

AMAZING COINCIDENCE: In representation theory, a connected real simple Lie group has discrete series representations if and only if it contains a compact real torus which is a Cartan subgroup

There is a beautiful emerging interaction between Mumford-Tate groups and representation theory

MUMFORD-TATE DOMAINS

Assume we have G connected linear algebraic group defined over $\mathbf{Q}$, T, $\phi\colon S^1 \to T$ giving a polarized Hodge structure to $\mathbf{g}$

$\mathbf{g} = T_eG$ has the structure of a vector space over $\mathbf{Q}$ lying inside it, and this allows us to choose a lattice Λ

The **centralizer in G of ϕ** is

$Z_G(\phi) = \{g \in G \mid g \circ \phi \circ g^{-1} = \phi\}$

$c_g(\phi) = g \circ \phi \circ g^{-1}\colon S^1 \to G$ also gives a polarized Hodge structure on $\mathbf{g}$ for all $g \in G$

$D = G/Z_G(\phi)$ is the space of all polarized Hodge structures on $\mathbf{g}$ obtained by conjugates of ϕ

D is called a **Mumford-Tate domain**

$c_g(\phi)$ has Mumford-Tate group contained in G

We let $\Gamma = \{g \in G \mid g(\Lambda) = \Lambda\}$, or some other discrete subgroup like this

$\Gamma \backslash D$ is the arithmetic quotient of the Mumford-Tate domain D

ROOTS

G, T as above, $\mathbf{t}$ the Lie algebra of T, $T = \mathbf{t}/L$ for a lattice $L \cong \mathbf{Z}^r$

Because the elements of $\mathbf{t}$ commute and are simultaneously diagonalizable, the eigenvalues of $\mathbf{t}$ acting on $\mathbf{g}$ are 0, with multiplicity r, and maps $\alpha\colon L \to \mathbf{Z}$

The set of α's which occur with non-zero eigenspace is called the **roots** of G, denoted Φ

It is a fact that these occur with multiplicity 1, i.e the mutual eigenspace $\mathbf{g}^\alpha$ has dimension 1

One can see that bracketing with elements of $\mathbf{t}$ stabilizes $\mathbf{k}$, $\mathbf{p}$ and thus $\mathbf{g}^\alpha$ is contained in either $\mathbf{k}$ or $\mathbf{p}$; these are called the **compact roots** Φ_c and the **non-compact roots** Φ_{nc}

Specifying a continuous homomorphism $\phi\colon S^1 \to T$ is the same as giving an element $L_\phi \in L$

The eigenspaces for ϕ are

$\mathbf{g}^j = \bigoplus_{\{\alpha\in\Phi | \alpha(L_\phi)=j\}} \mathbf{g}^\alpha$ for $j \neq 0$, and

$\mathbf{g}^0 = \mathbf{t} \oplus \bigoplus_{\{\alpha\in\Phi | \alpha(L_\phi)=0\}} \mathbf{g}^\alpha$

ADJOINT POLARIZED HODGE STRUCTURES IN TERMS OF ROOTS

Using the same notation, let R be the subgroup of $L^\vee = \mathrm{Hom}(L, \mathbf{Z})$ generated by the roots

One can show that there is a group homomorphism

$\Psi\colon R \to \mathbf{Z}/4\mathbf{Z}$ that satisfies

$\Psi(\alpha) = 0$ for $\alpha \in \Phi_c$

$\Psi(\alpha) = 2$ for $\alpha \in \Phi_{nc}$

We may think of $L_\phi \in L$ as giving a map $L_\phi\colon R \to \mathbf{Z}$

THEOREM: The condition that ϕ gives a polarized Hodge structure on g is that $L_\phi \equiv \Psi (\mathrm{mod}\ 4)$

Comment: The reason Ψ is a group homomorphism is that

$[\mathbf{k}, \mathbf{k}] \subset \mathbf{k}$, $[\mathbf{k}, \mathbf{p}] \subset \mathbf{p}$, $[\mathbf{p}, \mathbf{p}] \subset \mathbf{k}$

So sum of two compact roots is compact, sum of compact root and non-compact root is non-compact, etc.

COMPLEXIFICATION OF G

If G is a linear algebraic group defined over $\mathbf{Q}$, we may look at the solutions over $\mathbf{C}$ rather than over $\mathbf{R}$–this gives us the **complexification of** G, denoted $G_{\mathbf{C}}$

The Lie algebra of $G_{\mathbf{C}}$ is the complexification $\mathbf{g}_{\mathbf{C}}$ of $\mathbf{g}$

The g^{α} are contained in $\mathbf{g}_{\mathbf{C}}$

HODGE FILTRATION AND PARABOLIC SUBGROUPS

Recall the Hodge filtration, which in our situation is

$F^j\mathbf{g} = \bigoplus_{j' \geq j} \mathbf{g}^{2j'}$

Let $Q = \{g \in G_{\mathbf{C}} \mid \mathrm{Ad}_g(F^j) = F^j \text{ for all } j\}$

Q turns out to be what is called a **parabolic subgroup** of $G_{\mathbf{C}}$

The Lie algebra $\mathbf{q}$ of Q is

$\mathbf{q} = \bigoplus_{j \geq 0} \mathbf{g}^j = F^0\mathbf{g}$

WHICH ϕ's GIVE THE SAME MT DOMAIN

In $D^\vee = G_{\mathbf{C}}/Q$, at

$x = g_0 Q$, the isotropy group is

$Q_x = \{g \in G_{\mathbf{C}} \mid gx = x\} = g_0 Q g_0^{-1}$

Now $D = \{g_0 Q \mid g_0 \in G\} \subseteq D^\vee$

We get the same Mumford-Tate domain for different ϕ's for a given G if and only if the Q's are conjugate by G

VARIATION OF HODGE STRUCTURE FOR MUMFORD-TATE GROUPS

Let $D^\vee = G_{\mathbf{C}}/Q$, the **generalized flag variety**

We have a natural inclusion

$D \subseteq D^\vee$

If $x = \mathrm{Ad}_g(\phi)$, $g \in G$, then

$T_x D \cong T_x D^\vee \cong \mathbf{g}_{\mathbf{C}}/\mathrm{Ad}_g(\mathbf{q})$

The infinitesimal period relation says that the tangent space to any geometric family takes F^p to F^{p-1} in the earlier notation, or that it takes $\mathbf{g}^j$ to $\mathbf{g}^{j-2}$

This translates into the statement:

Mumford-Tate version of Griffiths transversality: The possible tangent spaces to geometric families with Mumford-Tate group contained in G at x is

$\mathrm{Ad}_g(\bigoplus_{j\geq -2} \mathbf{g}^j)/\mathbf{q} \subseteq \mathbf{g}_{\mathbf{C}}/\mathbf{q}$

which is isomorphic to

$\mathrm{Ad}_g(\mathbf{g}^{-2})$

Note that different ϕ's can give the same Mumford-Tate domain, but have a different infinitesimal period relation

MUMFORD-TATE DOMAINS GIVE INTERESTING DIFFERENTIAL SYSTEMS

Note that $[\mathbf{g}^j, \mathbf{g}^{j'}] \subseteq \mathbf{g}^{j+j'}$

In particular, $[\mathbf{g}^{-2}, \mathbf{g}^{-2}] \subseteq \mathbf{g}^{-4}$

This implies via the Frobenius condition for integrability that the tangent space to a geometric family at ϕ must be an abelian subalgebra $\mathbf{a} \subseteq \mathbf{g}^{-2}$

The differential system defined at each $x = c_g(\phi) \in D$, $g \in G$, defined by

$\mathrm{Ad}_g(\mathbf{g}^{-2})$

can have non-trivial brackets, and give a geometrically interesting differential system

SPECIAL POINTS IN MUMFORD-TATE DOMAINS

A point x in a Mumford-Tate domain $D = G/H$ will have its Mumford-Tate group contained in G.

Equality need not hold. As we move in D, the Hodge structure or one of its tensors may pick up additional Hodge classes

Additional Hodge classes may reduce the size of the Mumford-Tate group

HODGE STRUCTURES OF CM TYPE

Hodge structures whose Mumford-Tate group is a torus are said to be of **CM type**

CM stands for "complex multiplication"

Mumford-Tate domains have lots of points of CM type

Hodge structures of CM type are associated with number fields of a certain type

For example, for elliptic curves

$E = \mathbf{C}/\Lambda$, $\Lambda = \mathbf{Z} \oplus \lambda\mathbf{Z}$

then $H^1(E)$ is of CM type if and only if $\mathbf{Q}(1, \lambda) \cong \mathbf{Q}(\sqrt{-d})$ for some $d \in \mathbf{Z}^+$

BEYOND SHIMURA VARIETIES

There is a class of Mumford-Tate domains arising mainly from $H^1(X)$'s, i.e. Hodge structures of weight one, where D is a **Hermitian symmetric domain**

In this case, $\Gamma\backslash D$ has lots of sections of homogeneous line bundles and an arithmetic structure

This is the case of **Shimura varieties**

Mostly, when we look at $H^k(X)$ for $k > 1$, we are not in this case

There are lots of reasons for looking at these higher $H^k(X)$'s.

They come up in studying algebraic cycles of codimension ≥ 2

THE NON-SHIMURA CASE

While $\Gamma\backslash D$ in the non-Shimura case tends not have sections of homogeneous line bundles

Instead, what we have is lots of cohomology of homogeneous line bundles on $\Gamma\backslash D$

Very interesting work of Carayol for $G = SU(2,1)$ points the way to getting an arithmetic structure on this cohomology

The Hodge structures of CM type are expected to play a role in getting an arithmetic structure in general

There are interesting interactions with the Langlands program

FURTHER DIRECTIONS

There are a lot of interesting areas for further research:

(1) The closure of Mumford-Tate domains D in $D^\vee$ contains various **non-open G-orbits**, studied by Kerr-Pearlstein and also G-Griffiths

(2) The relationship of these orbits to compactifications of Hodge structures of families of varieties, i.e. **Limit Mixed Hodge structures**, is very rich, especially the relationship to the **Kato-Usui compactification**

(3) The geometry of the G-orbits is subtle, for example their **Levi forms** turn out to be interesting

(4) The **arithmetic of the** $\Gamma\backslash D$ remains mysterious and intriguing

(5) The **interaction with representation theory** is proving interesting