Groups acting on affine buildings and their boundaries

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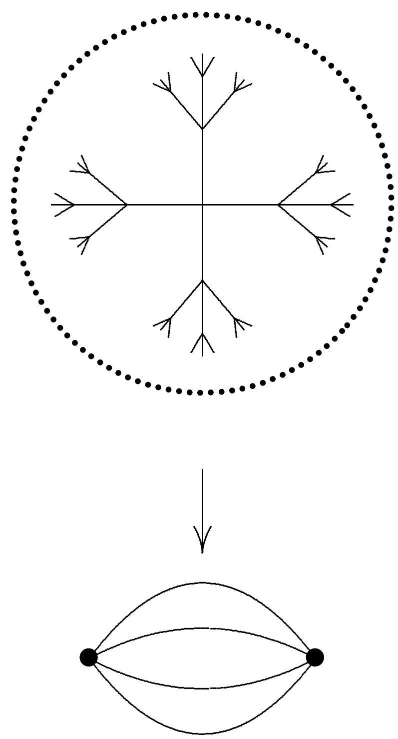
Example

X : A finite connected graph.

$\tilde{X}$: The universal covering space (a tree).

$\partial\tilde{X}$: The boundary of $\tilde{X}$.

$\tilde{X}$ and $\partial\tilde{X}$



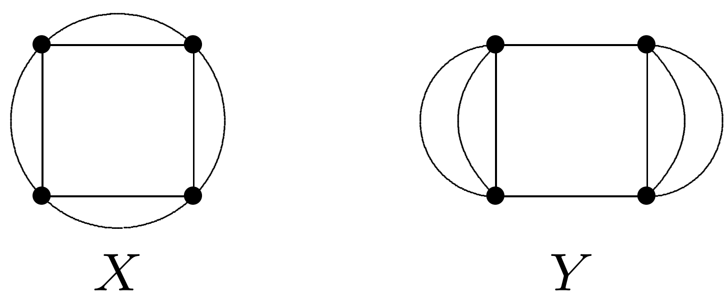
X

Let $\Gamma = \pi(X)$, the fundamental group of X.
Γ is a free group which acts freely on $\widetilde{X}$ and

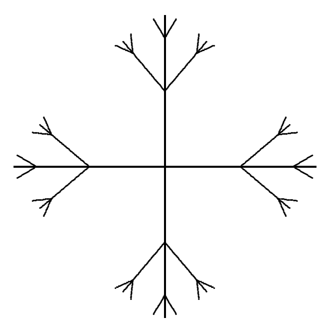
$$\Gamma\backslash\widetilde{X} = X.$$

Non-rigidity

The graphs



have the same universal covering Δ :

4-homogeneous tree 

isomorphic fundamental groups:

$$\Gamma_X \cong \Gamma_Y \cong F_5 \qquad \text{(free group of rank 5)}$$

Γ_X, Γ_Y are **not** conjugate as subgroups of Aut(Δ), since $X \not\cong Y$.

Continuous Analogue

The tree is a combinatorial analogue of the Poincaré upper half-plane

$$\widetilde{X} = \{z \in \mathbb{C} : \Im z > 0\}.$$

$G = \mathsf{PSL}_2(\mathbb{R}) = \mathsf{SL}_2(\mathbb{R})/\{-I, I\}$ acts on $\widetilde{X}$ via

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : z \mapsto \frac{\alpha z + \beta}{\gamma z + \delta}.$$

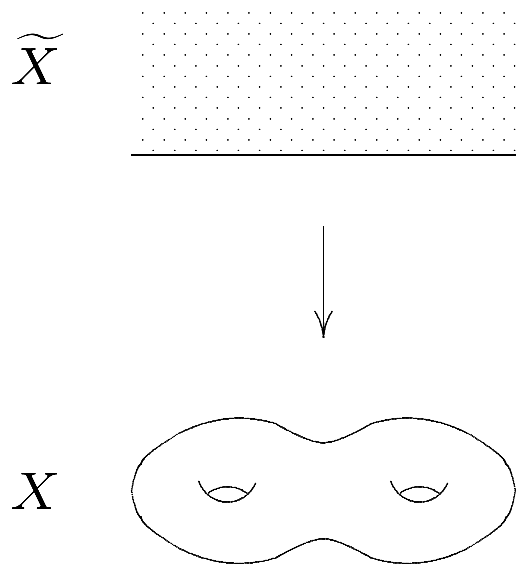
G acts transitively on $\widetilde{X}$, and on the boundary

$$\partial \widetilde{X} = \mathbb{R} \cup \{\infty\}\,.$$

A **cocompact lattice** in G is a discrete subgroup Γ of G with G/Γ compact.

A torsion free cocompact lattice Γ is the fundamental group of the Riemann surface

$$X = \Gamma \backslash \widetilde{X}.$$



Γ acts on $\widetilde{X}$, but there is **non-rigidity**: there are torsion free cocompact lattices

$$\Gamma \cong \Gamma'$$

which are **not** be conjugate in $\mathrm{PSL}_2(\mathbb{R})$.

Now replace $\mathbb{R}$ by $\mathbb{Q}_p$, p prime . . .

The p-adic field $\mathbb{Q}_p$

For $p \geq 2$ prime, $\mathbb{Q}_p$ is the field of formal sums :

$$x = a_j p^j + \cdots + a_0 + a_1 p + a_2 p^2 + \cdots ,$$

where each $a_i \in \{0, 1, \cdots, p-1\}$ and $a_j \neq 0$.

$$|x| = p^{-j} \qquad \textit{discrete valuation}$$

The p-adic integers

$$\begin{aligned} \mathbb{Z}_p &= \{x \in \mathbb{Q}_p : |x| \leq 1\} \\ &= \text{ set of sums with } j \geq 0 \\ &= \overline{\mathbb{Z}} \qquad \textit{a compact open subring of } \mathbb{Q}_p \end{aligned}$$

Note : $\mathbb{Z}_p \supset p\mathbb{Z}_p$.

The tree of $\mathrm{PGL}_2(\mathbb{Q}_p)$

Replace $\mathbb{R}$ by $\mathbb{Q}_p$, p prime. Let

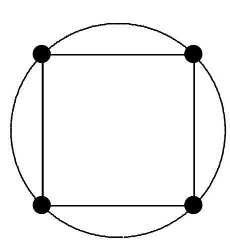
$$G = \mathrm{PGL}_2(\mathbb{Q}_p) = \mathrm{GL}_2(\mathbb{Q}_p)/\mathbb{Q}_p^\times.$$

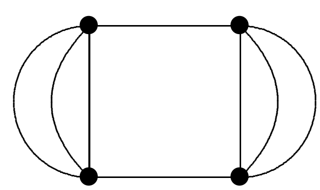
G acts on a homogeneous tree Δ of degree $p+1$. (A 1-dimensional **affine building**.)

If $\Gamma < G$ is a torsion free cocompact lattice, then Γ is a free group which acts on Δ.

The graph $X = \Gamma\backslash\Delta$ has universal cover Δ and fundamental group Γ.

Can choose $\Gamma_X, \Gamma_Y < \mathrm{PGL}_2(\mathbb{Q}_3)$ with quotient graphs





X Y

The building Δ of $\mathrm{PGL}_n(\mathbb{Q}_p)$

A lattice $L \subseteq \mathbb{Q}_p^n$ is a free $\mathbb{Z}_p$-submodule of rank n.

$$L = \mathbb{Z}_p v_1 + \mathbb{Z}_p v_2 + \cdots + \mathbb{Z}_p v_n,$$

where $(v_1, \ldots, v_n)$ is a basis of $\mathbb{Q}_p^n$.

Equivalence relation :

$$L_1 \sim L_2 \quad \Longleftrightarrow \quad L_1 = aL_2, \qquad a \in \mathbb{Q}_p^\times$$

An equivalence class $[L]$ is a **vertex** of Δ.

An **edge** of Δ is $([L_1], [L_2])$ where

$$L_1 \supset L_2 \supset pL_1.$$

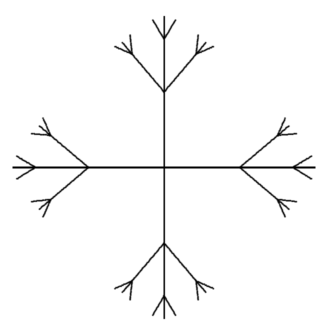
$\mathrm{GL}_n(\mathbb{Q}_p)$ acts transitively on the set of bases $(v_1, \ldots, v_n)$ of $\mathbb{Q}_p^n$.

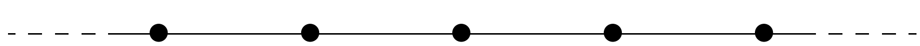
$\therefore G = \mathrm{PGL}_n(\mathbb{Q}_p)$ acts transitively on vertices.

Properties

- Δ is a topologically contractible simplicial complex of dimension $n-1$.
- Δ is a union of **apartments**: flat subcomplexes isomorphic to a simplicial tessellation of euclidean space.

The case $n = 2$: Δ is a tree





An apartment in a tree

The building Δ of $\mathrm{PGL}_3(\mathbb{Q}_p)$

A maximal simplex is a triangle

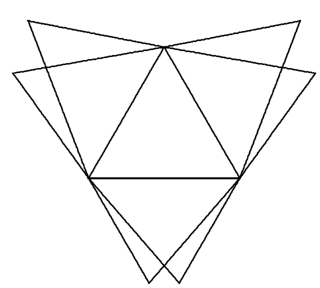
$$([L_1],[L_2],[L_3])$$

where:

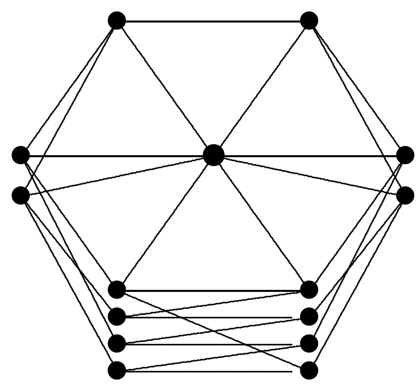
$$\begin{aligned} L_1 &= \mathbb{Z}_p v_1 + \mathbb{Z}_p v_2 + \mathbb{Z}_p v_3 \\ L_2 &= p\mathbb{Z}_p v_1 + \mathbb{Z}_p v_2 + \mathbb{Z}_p v_3 \\ L_3 &= p\mathbb{Z}_p v_1 + p\mathbb{Z}_p v_2 + \mathbb{Z}_p v_3 \end{aligned}$$

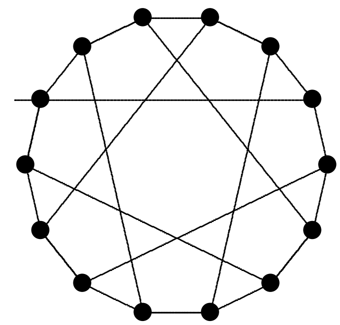
Note : $\mathbb{Z}_p \supset p\mathbb{Z}_p \Longrightarrow L_1 \supset L_2 \supset L_3 \supset pL_1$

Each edge of Δ lies on $p+1$ triangles.

$p = 2$: 

The neighbours of a vertex ($p = 2$)



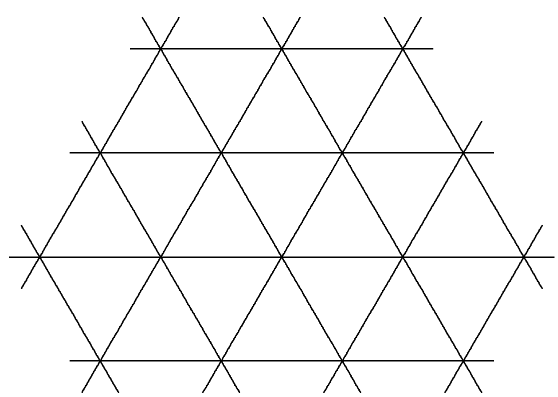


7 point
projective plane

On the left: a ball of radius one.

On the right: the graph obtained by deleting the centre of the ball and all the edges connected to it.

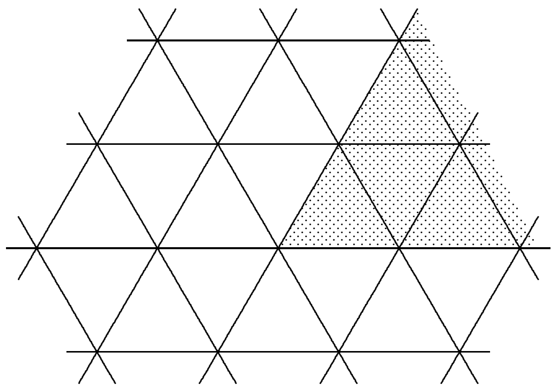
Δ is a union of apartments : flat subcomplexes isomorphic to a tessellation of $\mathbb{R}^2$ by equilateral triangles.



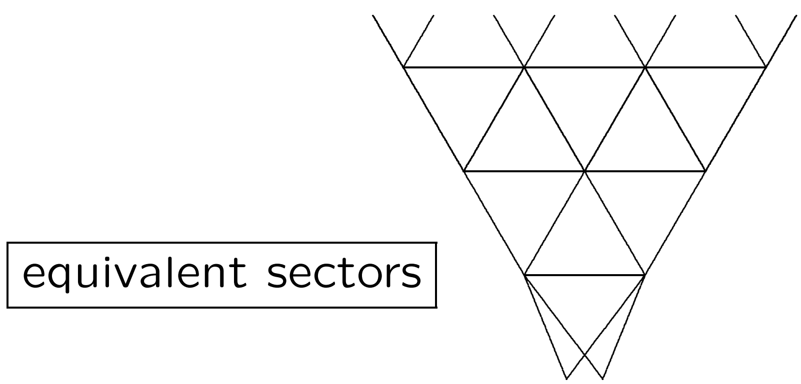
$\widetilde{A}_2$ Coxeter complex

The boundary $\partial\Delta$

A *sector* is a simplicial cone in some apartment.



Two sectors are *equivalent* if they contain a common subsector.



Boundary points are equivalence classes of sectors.

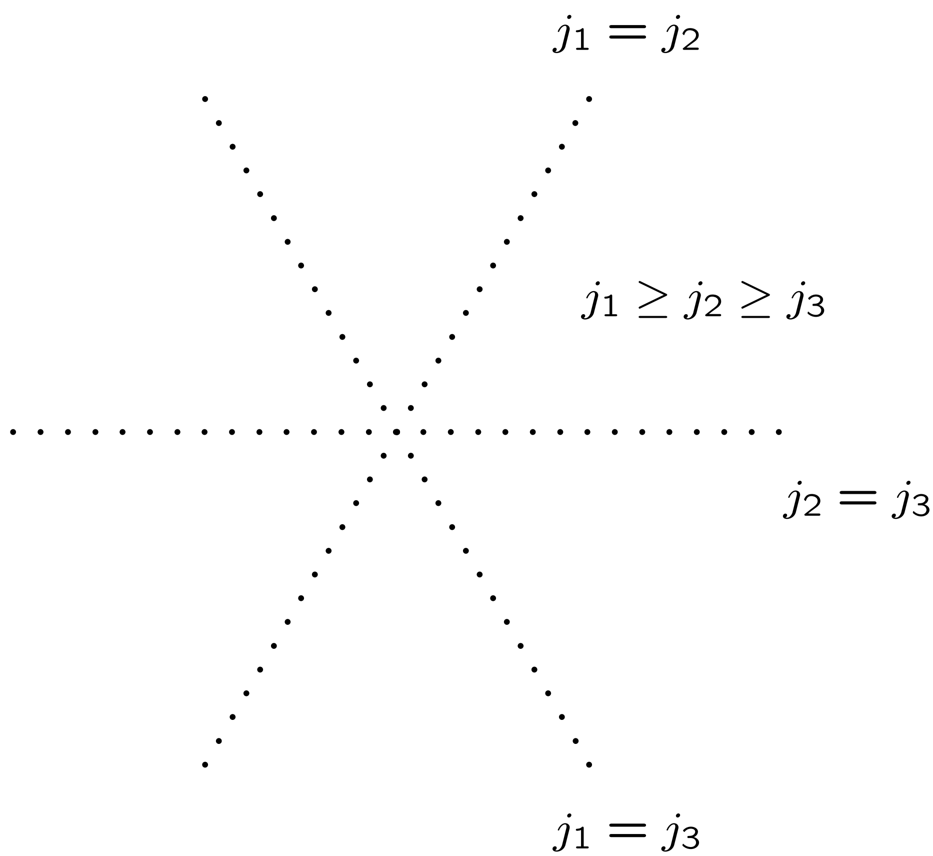
Fix a basis (v_1, v_2, v_3) of $\mathbb{Q}_p^3$. Then

$$[p^{j_1}\mathbb{Z}_p v_1 + p^{j_2}\mathbb{Z}_p v_2 + p^{j_3}\mathbb{Z}_p v_3]$$

are the vertices of an apartment A.

Vertices with $j_1 \geq j_2 \geq j_3$ define a sector with basepoint

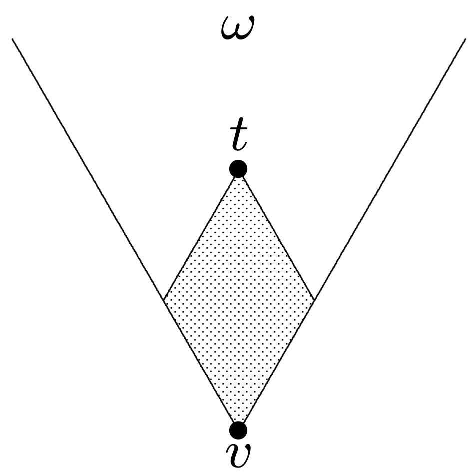
$$[\mathbb{Z}_p v_1 + \mathbb{Z}_p v_2 + \mathbb{Z}_p v_3].$$



The apartment is a union of $3! = 6$ sectors, corresponding to permutations of (v_1, v_2, v_3).

Topology on the boundary

Fix a vertex $v \in \Delta$. Given a vertex t in Δ, let Ω_t be the set of all $\omega \in \partial\Delta$ represented by a sector containing $\mathrm{conv}\{v,t\}$.

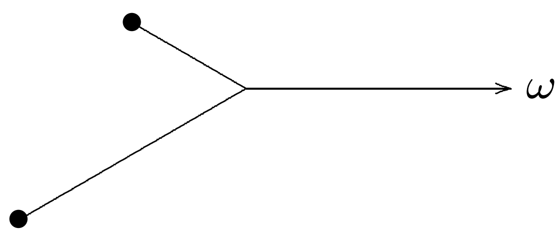


Ω_t is a basic open set for a totally disconnected compact topology on $\partial\Delta$.

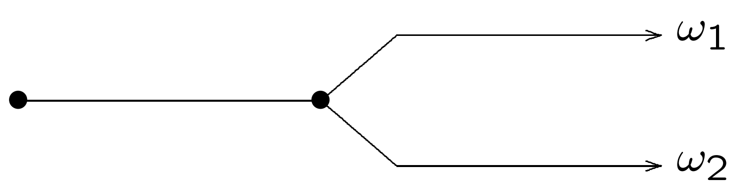
$G = \mathrm{PGL}_3(\mathbb{Q}_p)$ acts transitively on $\partial\Delta$.

The tree case

Equivalent Sectors

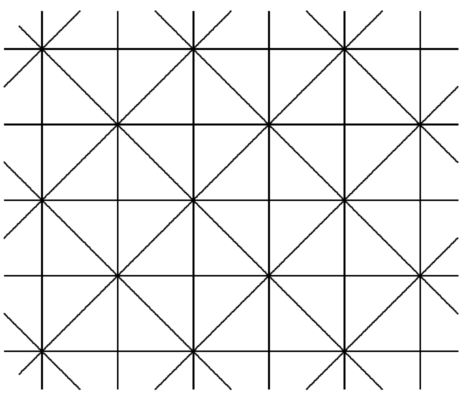


Neighbouring Boundary Points

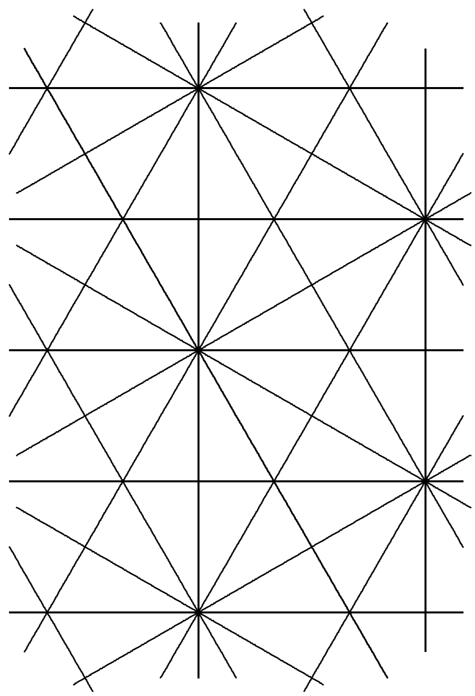


A semisimple algebraic group G over $\mathbb{Q}_p$ acts on its affine building.

Other 2-dimensional affine buildings



$\widetilde{B}_2$ apartment



$\widetilde{G}_2$ apartment

$\widetilde{B}_2$ comes from the group $SO_5(Q)$:
elements of SL_5 preserving the quadratic form

$$Q = x_1x_5 + x_2x_4 + x_3^2.$$

$\widetilde{G}_2$ comes from an exceptional group.

Rigidity

Let $\mathbb{F}_1, \mathbb{F}_2$ be nonarchimedean local fields.
e.g. $\mathbb{Q}_p$ or a finite extension of $\mathbb{Q}_p$.

For $i = 1, 2$, consider

- Δ_i : the affine building of $\mathrm{PGL}_n(\mathbb{F}_i)$,
- $\Gamma_i < \mathrm{PGL}_n(\mathbb{F}_i)$: a torsion free lattice.

If $n \geq 3$, then

- $\Delta_1 \cong \Delta_2 \;\Rightarrow\; \mathbb{F}_1 \cong \mathbb{F}_2$;
- $\Gamma_1 \cong \Gamma_2 \quad \Rightarrow \quad \begin{cases} \Delta_1 \cong \Delta_2\,, \\ \\ \Gamma_1 \backslash \Delta_1 \cong \Gamma_2 \backslash \Delta_2\,. \end{cases}$

(J.Tits ; G. Mostow, G. A. Margulis).

Both statements fail for $n = 2$.

e.g. The 4-homogeneous tree Δ is the affine building of both $\mathrm{PGL}_2(\mathbb{Q}_3)$, $\mathrm{PGL}_2(\mathbb{Q}_3(\sqrt{3}))$.*
Both groups contain torsion free lattices $\Gamma_1 \cong \Gamma_2 = F_5$, with $\Gamma_1 \backslash \Delta \ncong \Gamma_2 \backslash \Delta$.

*$\mathbb{Q}_3(\sqrt{3})$ could be replaced by any other totally ramified extension of $\mathbb{Q}_3$.

Fundamental Idea. (G. Mostow.)
Any isomorphism

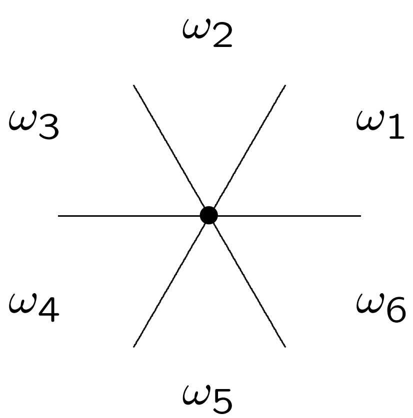
$$f : \Gamma_1 \to \Gamma_2$$

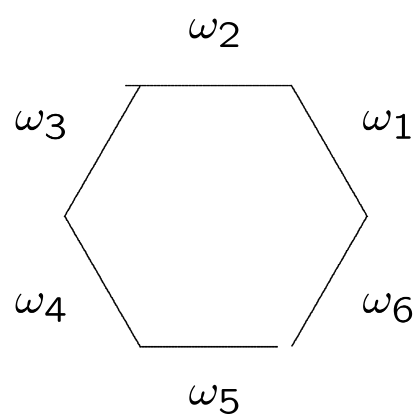
induces an equivariant isomorphism

$$f_* : \partial\Delta_1 \to \partial\Delta_2.$$

Rigidity and higher rank boundaries.

Example: PGL_3
An apartment in Δ has 6 boundary points.





The ω_i are also the maximal simplices of the **spherical building** at infinity. An **apartment** is a hexagon.