Operator algebras, boundaries of buildings and K-theory

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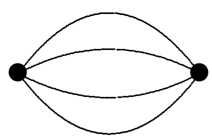
Spaces that arise in analysis are often pathological and cannot be studied by classical geometric methods. Consider instead an associated **algebraic** object.

For example, the topology of a "good" space S is completely determined by the commutative algebra

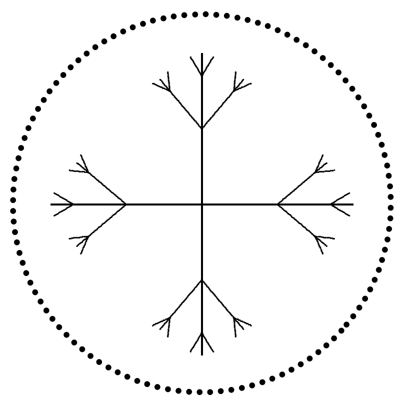
$$C(S) = \{f : S \to \mathbb{C} \mid f \text{ is continuous}\}.$$

If S is a "bad" space, replace $C(S)$ by a **non-commutative** algebra.

Example: A finite connected graph X, with all vertices of degree > 2. The universal covering tree Δ has boundary $\partial\Delta$.



X



Δ and $\partial\Delta$

Let $\Gamma = \pi(X)$, the fundamental group of X. Γ is a free group which acts freely on Δ and

$$\Gamma\backslash\Delta = X$$

Γ also acts on $\partial\Delta$, but this action is "bad": $\Gamma\backslash\partial\Delta$ is not Hausdorff.

Γ acts on $C(\partial\Delta)$:

$$\gamma(f)(\omega) = f(\gamma^{-1}\omega)$$

Study the "bad" action by forming the **crossed product C^*-algebra**:

$$\begin{aligned}\mathcal{A}(\Gamma) &= C(\partial\Delta) \rtimes \Gamma \\ &= C^*\langle \Gamma \cup C(\partial\Delta)\ ;\ \gamma(f) = \gamma f \gamma^{-1}\rangle\end{aligned}$$

Here

$C(\partial\Delta) \subset \mathcal{A}(\Gamma)$	an abelian subalgebra
$\Gamma \subset \mathcal{A}(\Gamma)$	a group of unitaries

$\mathcal{A}(\Gamma)$ is generated (as a C^*-algebra) by finite sums

$$\sum f_i\gamma_i, \qquad f_i \in C(\partial\Delta),\ \gamma_i \in \Gamma$$

product : $\quad f_1\gamma_1 \cdot f_2\gamma_2 = f_1\gamma_1(f_2)\gamma_1\gamma_2$

involution : $\quad (f\gamma)^* = \gamma^{-1}(f^*)\gamma^{-1}$

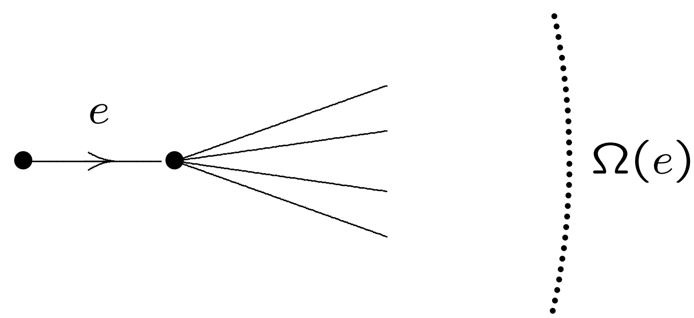
What about "good" actions?

Example : $\quad X = \Gamma\backslash\Delta$.

Answer : $\quad C_0(\Delta) \rtimes \Gamma \approx C(X)$.

Let $E = \{\text{oriented edges of } \Delta\}$.
(Each geometric edge has two orientations.)

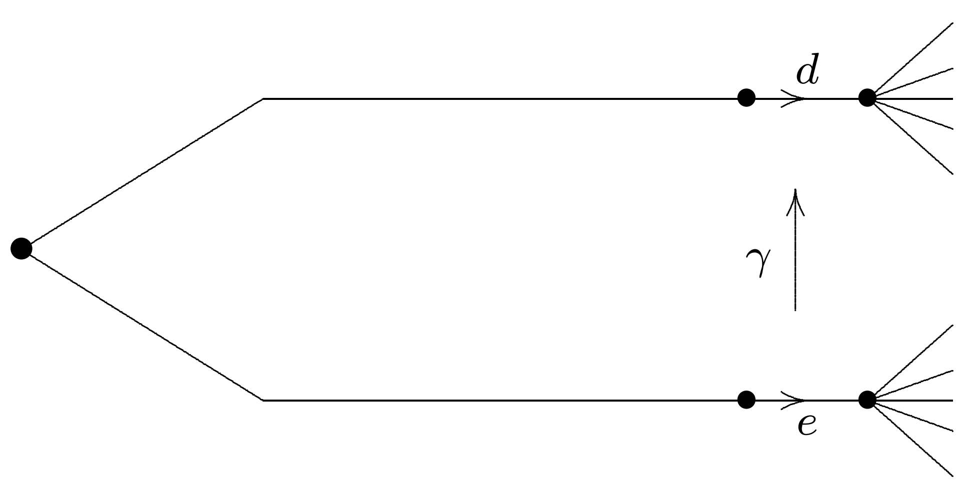
If $e \in E$, define a clopen subset $\Omega(e)$ of $\partial\Delta$



The indicator function $p_e \in C(\partial\Delta) \subset \mathcal{A}(\Gamma)$.

If $e \in E$ and $d = \gamma e$, where $\gamma \in \Gamma$, then define a **partial isometry**

$$s_{d,e} = \gamma p_e \, .$$



$$s_{d,e}^* s_{d,e} = p_e \qquad \text{initial projection}$$
$$s_{d,e} s_{d,e}^* = \gamma p_e \gamma^{-1}$$
$$= \gamma(p_e)$$
$$= p_d \qquad \text{final projection}$$

Therefore $p_d \sim p_e$

(Murray–von Neumann equivalence)

Facts :

(1) $\mathcal{A}(\Gamma)$ is simple, purely infinite and generated by the operators $s_{d,e}$, where $d \in \Gamma e$.
A **Cuntz-Krieger algebra.**

(2) $\mathcal{A}(\Gamma)$ is classified by the abelian group $K_0(\mathcal{A}(\Gamma))$ and [1], where

$$K_0(\mathcal{A}) = \{[p] : p \text{ is a nonzero idempotent in } \mathcal{A}\}$$

addition : $[p] + [q] = [p + q]$, if $pq = 0$,

zero element : $[p - p']$, where $p \sim p' < p$.

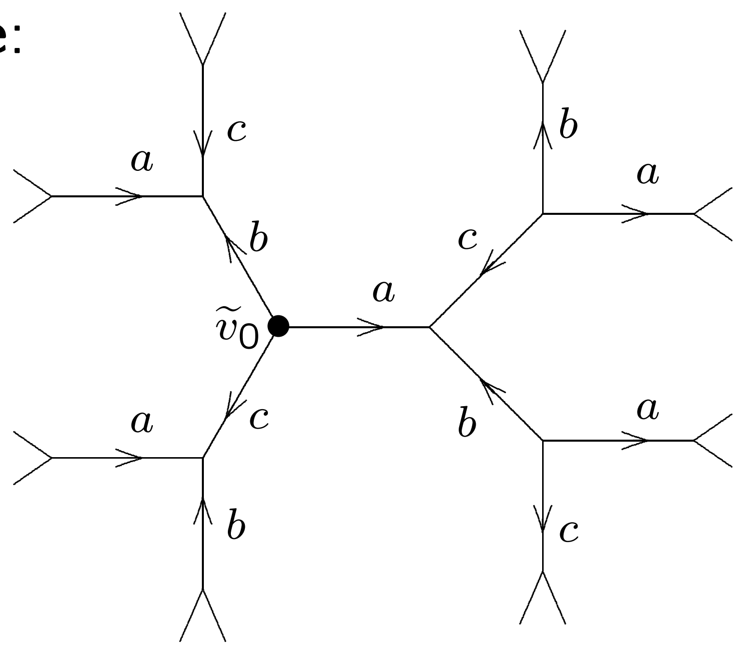
If $e \in E$, $[p_e] \in K_0(\mathcal{A}(\Gamma))$ depends only on Γe.

Let $A := \{\Gamma e : e \in E\}$, (*a finite alphabet*).

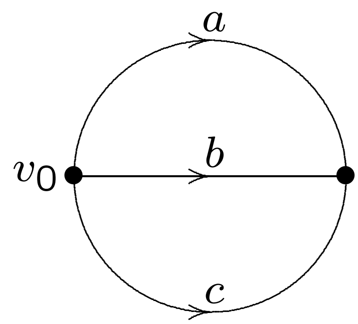
$$A \approx \{\text{directed edges of } X\}$$

Example:

Δ



X



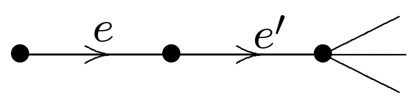
$A = \{a, \overline{a}, b, \overline{b}, c, \overline{c}\}$

For $a = \Gamma e \in A$, let $[a] := [p_e] \in K_0(\mathcal{A}(\Gamma))$.

These are **all** the generators for $K_0(\mathcal{A}(\Gamma))$.

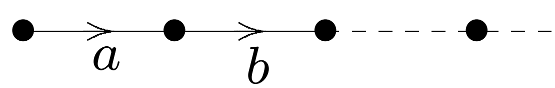
The idempotents p_e satisfy

$$p_e = \sum_{\substack{e' \in E \\ e \to e'}} p_{e'}$$



Define 0–1 matrix M, for a, $b \in A$ by

$M(a,b) = 1 \iff$



Relations:

$$[a] = \sum_{b \in A} M(a,b)[b]\,.$$

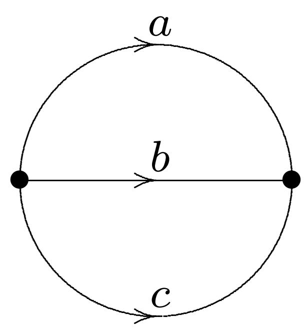
These are the **only** relations . . .

Theorem.

$$K_0(\mathcal{A}(\Gamma)) = \left\langle A \;\middle|\; a = \sum_{b \in A} M(a,b) b \right\rangle .$$

This is easily computed from the graph X:

Example:



generators: $\{a, \overline{a}, b, \overline{b}, c, \overline{c}\}$

relations:

$$\begin{aligned} a &= \overline{b} + \overline{c}, \quad b = \overline{c} + \overline{a}, \quad c = \overline{a} + \overline{b} \\ \overline{a} &= b + c, \quad \overline{b} = c + a, \quad \overline{c} = a + b \end{aligned}$$

Result:
$K_0(\mathcal{A}(\Gamma)) = \mathbb{Z}^r \oplus \mathbb{Z}/(r-1)\mathbb{Z}$, where r is the rank of Γ. The class $[1] \in K_0(\mathcal{A}(\Gamma))$ has order

$$r - 1 = -\chi(X)$$

(Euler characteristic)

Remark: It follows that $\mathcal{A}(\Gamma)$ depends only on Γ.

Special case:
Let p be prime and Γ a torsion free lattice in

$$G = \mathrm{PGL}_2(\mathbb{Q}_p)\,.$$

G acts on Δ (homogeneous tree of degree $p+1$),

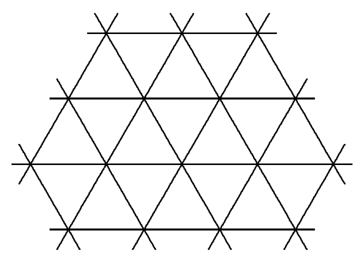
$X = \Gamma\backslash\Delta$ is a finite graph, $\Gamma = \pi(X)$ and

$$\chi(X) = -\frac{(p-1)}{2} \cdot \#\{\text{vertices of } X\}$$

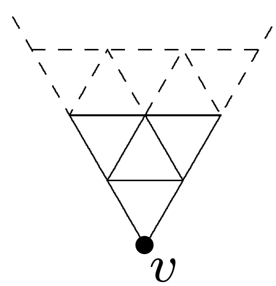
$\boxed{\mathsf{PGL}_3(\mathbb{Q}_p)}$

$G = \mathsf{PGL}_3(\mathbb{Q}_p)$ acts on its **building** of type $\widetilde{A}_2$, which is a topologically contractible 2-dimensional complex Δ.

Δ is a union of **apartments**: flat subcomplexes isomorphic to a tessellation of $\mathbb{R}^2$ by equilateral triangles.



The boundary $\partial\Delta$ is a compact totally disconnected space whose points correspond to sectors in Δ based at a fixed vertex v.



The boundary algebra $\mathcal{A}(\Gamma)$.

If Γ is a torsion free lattice in $\mathrm{PGL}_3(\mathbb{Q}_p)$ then Γ acts freely on Δ, the universal cover of the 2-dimensional complex $X = \Gamma\backslash\Delta$.

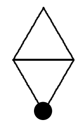
X is determined by Γ, by Strong Rigidity.

Γ acts on Δ, and on $\partial\Delta$. Define

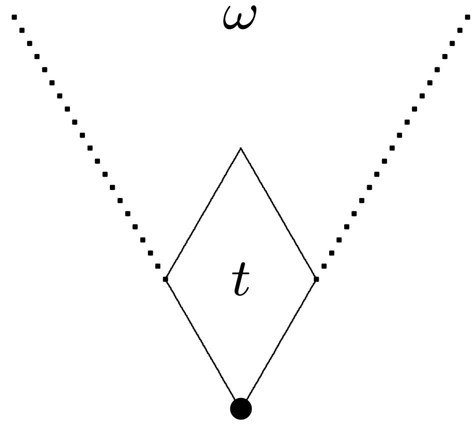
$$\mathcal{A}(\Gamma) := C(\partial\Delta) \rtimes \Gamma .$$

[Depends only on Γ.]

The algebras $\mathcal{A}(\Gamma) = C(\partial\Delta) \rtimes \Gamma$ are examples of **higher rank Cuntz-Krieger algebras** whose structure theory has been developed by G. Robertson and T. Steger (1998-2001).

Given a basepointed tile $t =$  in Δ,

let Ω_t be the set of all $\omega \in \partial\Delta$ such that



Let
$p_t =$ char. function of $\Omega_t \in C(\partial\Delta) \subseteq \mathcal{A}(\Gamma)$.

Then

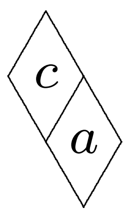
$$\gamma p_t \gamma^{-1} = p_{\gamma t}$$

so

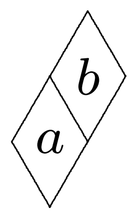
$$[p_t] \in K_0 \quad \text{depends only on } \Gamma t.$$

Let $A := \{\Gamma t : t \text{ a tile}\}$, (*a finite alphabet*).

Define 0–1 matrices M_1, M_2, for a, $b \in A$ by



$M_2(a, c) = 1$



$M_1(a, b) = 1$

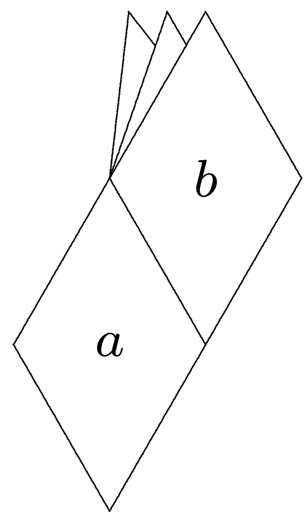
For $a = \Gamma t \in A$, let $[a] := [p_t] \in K_0(\mathcal{A}(\Gamma))$.

Relations:

- $[a] = \sum_{b \in A} M_1(a, b)[b]$;

- $[a] = \sum_{b \in A} M_2(a, b)[b]$.

These are the **only** relations . . .



$$[a] = \sum_{b \in A} M_1(a,b)[b]\,;$$

Let

$$C = \left\langle A \,\middle|\, a = \sum_{b\in A} M_j(a,b)b,\ j=1,2 \right\rangle .$$

Theorem. [G. Robertson, T. Steger, 2001]

$$K_0(\mathcal{A}(\Gamma)) = C \oplus \mathbb{Z}^{\mathrm{rank}(C)}.$$

Note : There are generators of K_0 **not** of the form $[a] = [p_t]$.

Theorem. $m.[\mathbf{1}] = 0$ in $K_0(\mathcal{A}(\Gamma))$, where

$$m = \gcd(3, p-1) \cdot \frac{(p^2-1)}{3} \cdot \#\{\text{vertices of } \Gamma\backslash\Delta\}$$

Strong numerical evidence suggests that the order of $[\mathbf{1}]$ is actually :

$$\frac{(p-1)}{\gcd(3,p-1)} \cdot \#\{\text{vertices of } \Gamma\backslash\Delta\}$$

Note:

$$\chi(\Gamma\backslash\Delta) = \tfrac{(p-1)(p^2-1)}{3} \cdot \#\{\text{vertices of } \Gamma\backslash\Delta\}.$$

Example: The simplest possible Γ has generators $x_0, x_1, \ldots, x_6$, and relations

$$\begin{cases} x_0x_1x_4, x_0x_2x_1, x_0x_4x_2, x_1x_5x_5, \\ x_2x_3x_3, x_3x_5x_6, x_4x_6x_6. \end{cases}$$

- Γ is a torsion free lattice in $\mathrm{PGL}_3(\mathbb{Q}_2)$.
- Γ acts transitively on vertices of Δ.

$$K_0(\mathcal{A}(\Gamma)) = (\mathbb{Z}/2\mathbb{Z})^2 \oplus \mathbb{Z}/3\mathbb{Z},$$
$$[\mathbf{1}] = 0$$

$\exists$ exactly 3 such $\Gamma < \mathrm{PGL}_3(\mathbb{Q}_2)$.
(Cartwright, Mantero, Steger, Zappa, 1993)

3 different groups $K_0(\mathcal{A}(\Gamma))$:

$\mathbb{Z}/3\mathbb{Z}$ $\quad (\mathbb{Z}/2\mathbb{Z})^2 \oplus \mathbb{Z}/3\mathbb{Z}$ $\quad (\mathbb{Z}/2\mathbb{Z})^4 \oplus \mathbb{Z}/3\mathbb{Z}$

Other affine buildings: the boundary algebras $\mathcal{A}(\Gamma)$ are again simple and purely infinite, but $K_0(\mathcal{A}(\Gamma))$ is harder to compute. However

Theorem. Let G be a semisimple Chevalley group over $\mathbb{Q}_p$. Let Γ be a lattice in G. Then $[1]$ has finite order in $K_0(\mathcal{A}(\Gamma))$.

If G is not type $\widetilde{E}_8$ or $\widetilde{F}_4$, and Γ is torsion free, then

$$\text{order of } [1] < \#\{\text{faces of } \Gamma\backslash\Delta\}.$$

Continuous Analogue: $\Gamma < \mathrm{PSL}_2(\mathbb{R})$, the fundamental group of a Riemann surface M.

Γ acts on the Poincaré upper half-plane

$$\mathfrak{H} = \{z \in \mathbb{C} : \Im z > 0\}.$$

and on $\partial\mathfrak{H} = \mathbb{R} \cup \{\infty\} = \mathbb{S}^1$. Let

$$\mathcal{A}(\Gamma) = C(\mathbb{S}^1) \rtimes \Gamma.$$

Fact: The class $[1] \in K_0(\mathcal{A}(\Gamma))$ has order $-\chi(M)$. (A. Connes; T. Natsume)

Question : Is $[1]$ always torsion for geometric boundary algebras?

Example: $G = \mathrm{PSL}_2(\mathbb{C})$ acts on hyperbolic 3-space and its boundary S^2.

If $\Gamma < G$ is a countable discrete subgroup then $[1]$ is **not** torsion in $K_0(\mathcal{A}(\Gamma))$ (A. Connes).

APPENDIX

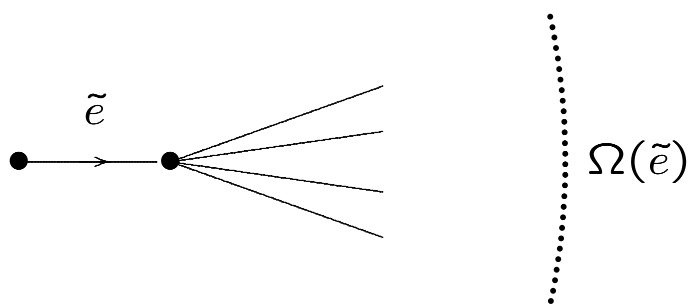
Proof of tree case: $-\chi(\Gamma).[1] = 0$ in $K_0(\mathcal{A}(\Gamma)$.

$\widetilde{E} = \widetilde{E}_+ \sqcup \widetilde{E}_- = \{\text{oriented edges of } \Delta\}$

$\widetilde{V} = \{\text{vertices of } \Delta\}$

E, V : oriented edges, vertices of $X = \Gamma \backslash \Delta$

If $\tilde{e} \in \widetilde{E}$, $\Omega(\tilde{e})$ is a clopen subset of $\partial\Delta$:



The indicator function $p_{\tilde{e}} \in C(\partial\Delta) \subset \mathcal{A}(\Gamma)$.

$[p_{\tilde{e}}] \in K_0(\mathcal{A}(\Gamma))$ depends only on $e = \Gamma\tilde{e} \in E$.

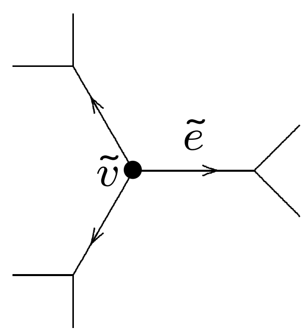
Reason: $p_{\gamma\tilde{e}} = \gamma \cdot p_{\tilde{e}} = \gamma p_{\tilde{e}} \gamma^{-1}$.

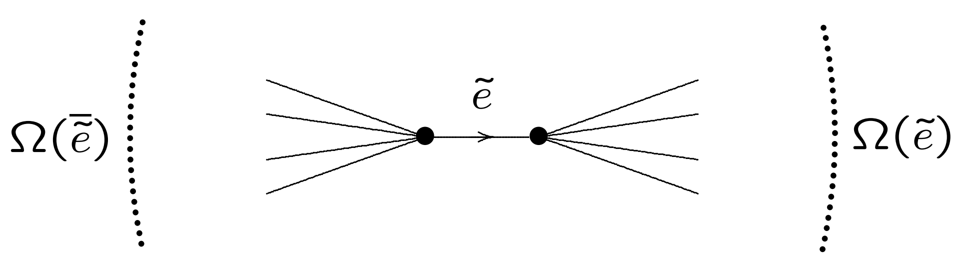
Therefore write $[e] = [p_{\tilde{e}}] \in K_0(\mathcal{A}(\Gamma))$.

The idempotents $p_{\tilde{e}}$ satisfy the following relations

$$\sum_{\substack{\tilde{e}\in\tilde{E} \\ o(\tilde{e})=\tilde{v}}} p_{\tilde{e}} = \mathbf{1}, \qquad \text{for } \tilde{v}\in\tilde{V}; \qquad (1a)$$

$$p_{\tilde{e}} + p_{\overline{\tilde{e}}} = \mathbf{1}, \qquad \text{for } \tilde{e}\in\tilde{E}. \qquad (1b)$$





The relations (1) project to the following relations in $K_0(\mathcal{A}(\Gamma))$.

$$\sum_{\substack{e\in E\\ o(e)=v}} [e] \;=\; [\mathbf{1}], \qquad \text{for } v \in V; \qquad (2a)$$

$$[e] + [\overline{e}] \;=\; [\mathbf{1}], \qquad \text{for } e \in E. \qquad (2b)$$

Since the map $e \mapsto o(e) : E \to V$ is surjective, the relations (2) imply that

$$\begin{aligned} n_V[\mathbf{1}] &= \sum_{v\in V} \sum_{\substack{e\in E\\ o(e)=v}} [e] = \sum_{e\in E} [e] \\ &= \sum_{e\in E_+} ([e] + [\overline{e}]) = \sum_{e\in E_+} [\mathbf{1}] \\ &= n_{E_+}[\mathbf{1}]. \end{aligned}$$

Therefore $(n_V - n_{E_+}).[\mathbf{1}] = 0$.

i.e $$\chi(X).[\mathbf{1}] = 0.$$