Analysis on Covering Spaces

A Survey

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The purpose of this lecture

To review ideas, methods and results in analysis on covering spaces, especially analysis on covering graphs over finite graphs.

1) Quick review of covering spaces

2) Twisted Laplacians and Kazhdan distance

3) Analysis on Cayley graphs (cogrowth, Ramanujan graphs, zeta functions of finitely generated groups)

5) Abel-Jacobi maps in graph theory

6) Large deviation asymptotics of heat kernels on periodic manifolds

1. Quick Review of Covering Spaces

Covering Spaces

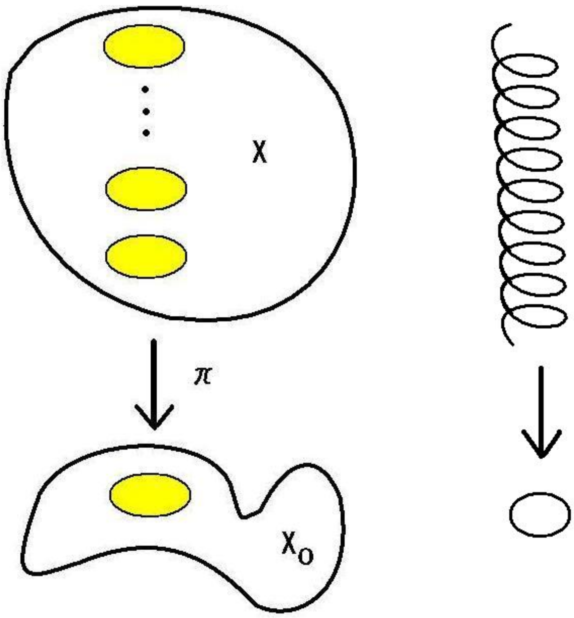
As spaces, we mainly treat manifolds and 1-dimensional cell complexes (graphs).

Roughly speaking, a covering map is a surjective map of spaces $\pi : X \longrightarrow X_0$ which preserves the local structure (topology, Riemannian metric, adjacency relation (and weights) of graphs).

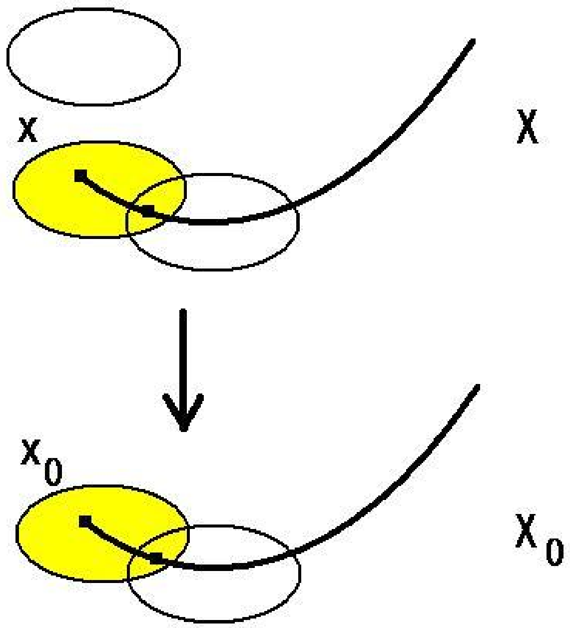
X is said to be a covering space over X_0.

In this lecture, the base space X_0 is supposed to be compact (thus in the cese of graphs, X_0 is supposed to be a finite graphs).

A schematic image of a covering map



Unique lifting of paths



Regular covering spaces

If a group Γ acts on a space freely and discontinuously, then the canonical map $\pi : X \longrightarrow \Gamma\backslash X = X_0$ is a covering map.

A covering map (space) obtained in this way is called a regular covering map (space) with covering transformation group Γ.

A regular covering space with abelian covering transformation group is called an abelian covering space.

Universal covering

Among all covering spaces over a fixed space X_0, there is a "maximal one", which is called the universal covering map and is characterized by simply connectedness.

The universal covering space over X_0 is a regular covering space whose covering transfromation group is the fundamental group $\pi_1(X_0)$.

As a set, $\pi_1(X_0)$ is the set of homotopy classes of loops in X_0 with a fixed base point.

Galois Theory for covering maps

A covering space X over X_0

$\Longleftrightarrow$

A subgroup Γ of $\pi_1(X_0)$

The correspondence is given as

$$X \Longrightarrow \Gamma = \pi_1(X)$$
$$\Gamma \Longrightarrow X = \Gamma\backslash\widehat{X_0},$$

where $\widehat{X_0}$ is the universal covering space over X_0.

A regular covering space X over X_0

$\Longleftrightarrow$

A normal subgroup Γ of $\pi_1(X_0)$.

This being the case, the factor group $G = \Gamma\backslash\pi_1(X_0)$ is the covering transformation group of $X \longrightarrow X_0$.

Abelian covering maps

Let $[\pi_1(X_0), \pi_1(X_0)]$ be the commutator group (the normal subgroup of $\pi_1(X_0)$ generated by elements of the form $[a, b] = aba^{-1}b^{-1}$),

Note $H_1(X_0, \mathrm{Z}) = [\pi_1(X_0), \pi_1(X_0)]\backslash\pi_1(X_0)$, the 1^{st} homology group of X_0 (Hurewitz).

Thus $X = [\pi_1(X_0), \pi_1(X_0)]\backslash\widehat{X_0}$ is the covering space whose covering transformation group is $H_1(X_0, \mathrm{Z})$.

This X is "maximal among all abelian covering spaces over X_0.

(Use the fact that $\Gamma\backslash\pi_1(X_0)$ is abelian if and only if Γ contains the commutator group $[\pi_1(X_0), \pi_1(X_0)]$).

An abelian covering space X over X_0

$\Longleftrightarrow$

A subgroup Γ of $\pi_1(X_0)$ with $[\pi_1(X_0), \pi_1(X_0)] \subset \Gamma$

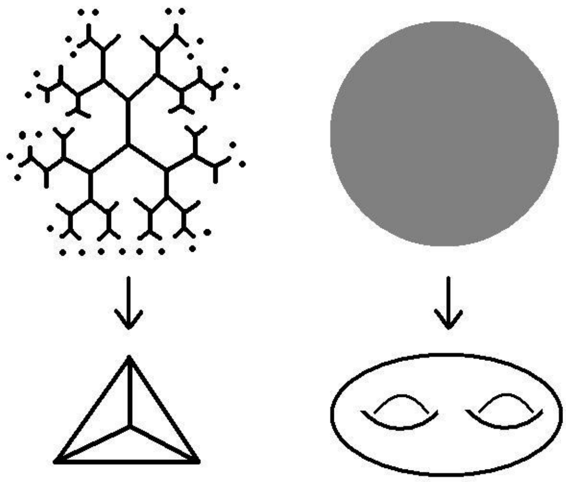
$\Longleftrightarrow$

A subgroup H of $H_1(X_0, \mathbb{Z})$

$\Longleftrightarrow$

A surjective homomorphism $H_1(X_0, \mathbb{Z}) \longrightarrow G$

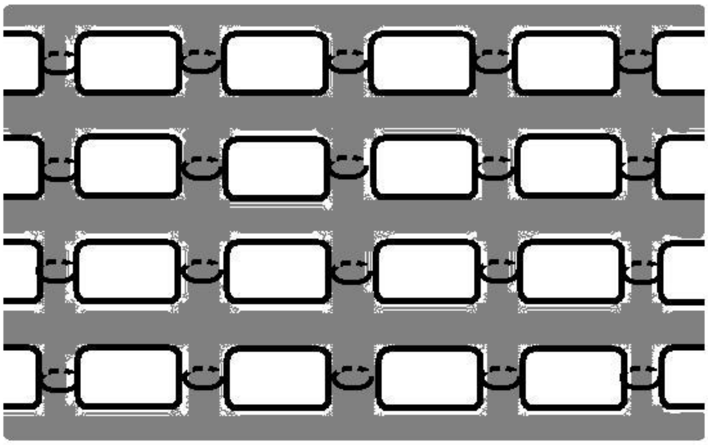
Example of Universal Covering Spaces



Since a simply connected graph is tree, the universal covering graph is a tree.

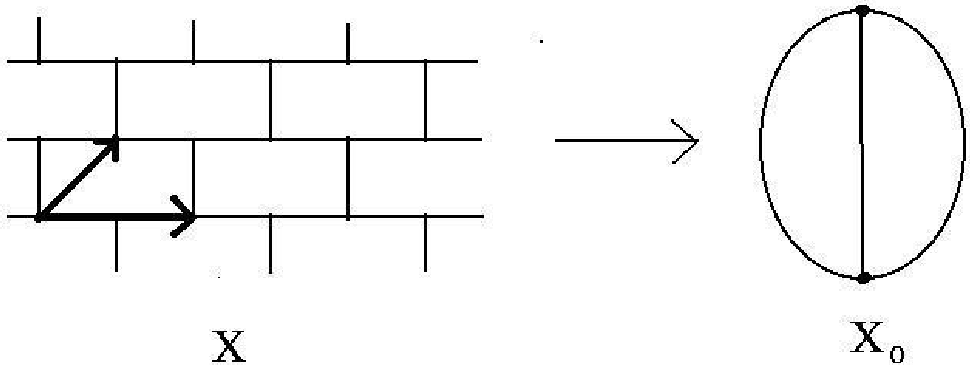
Abelian Covering Spaces

I)



This is an abelian covering surface over a closed surface of genus two.

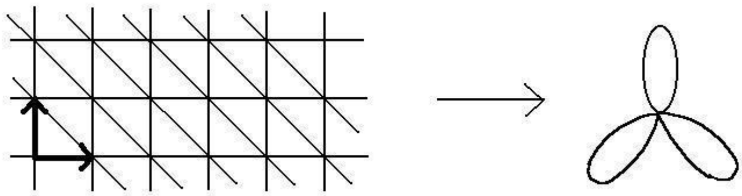
II)



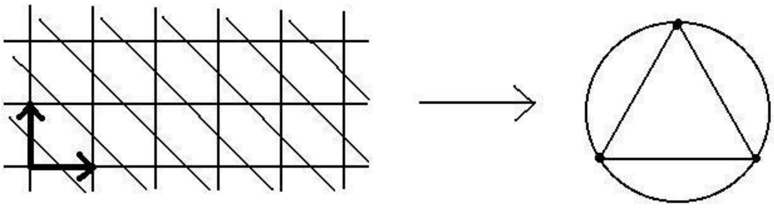
Hexagonal lattice

In general, a covering graph over a finite graph with free abelian covering transformation group is called a crystal lattice or topological crystal.

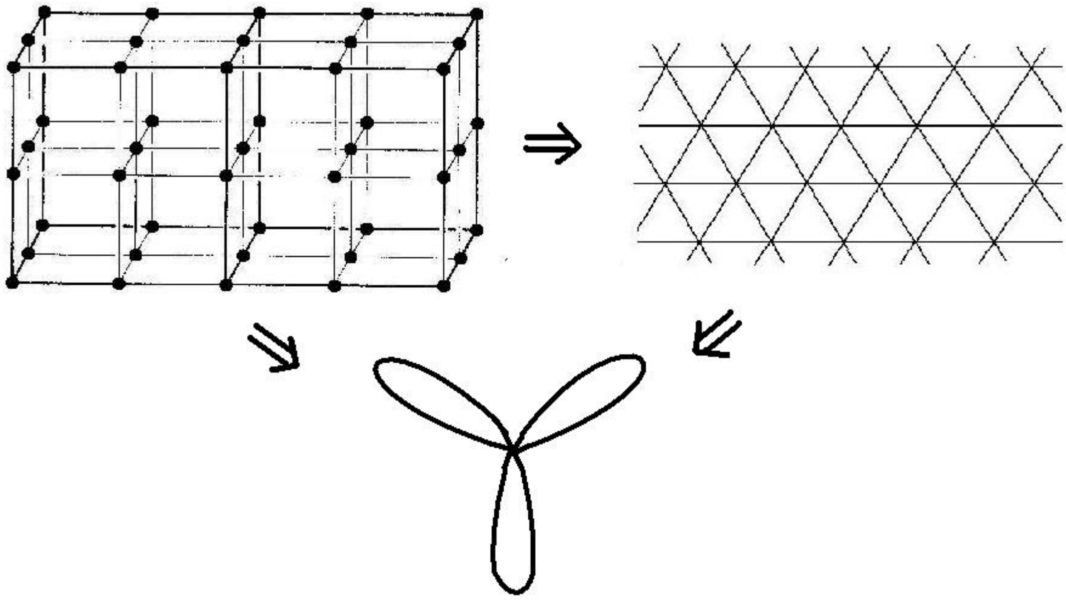
Triangular lattice



Kagome lattice



Kagome lattice

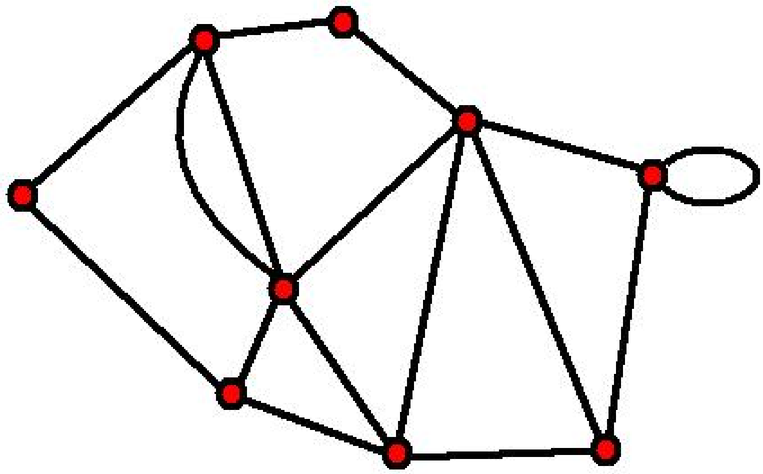


2. Quick Review of Graphs

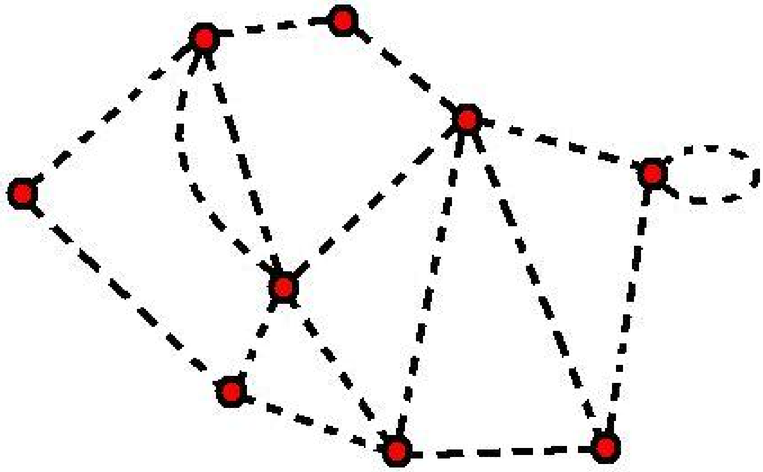
Grahps –Terminology–

A graph X is an abstract figure consisting of two kind of objects; say vertices and edges.

Denote as $X = (V, E)$, and "realize" as

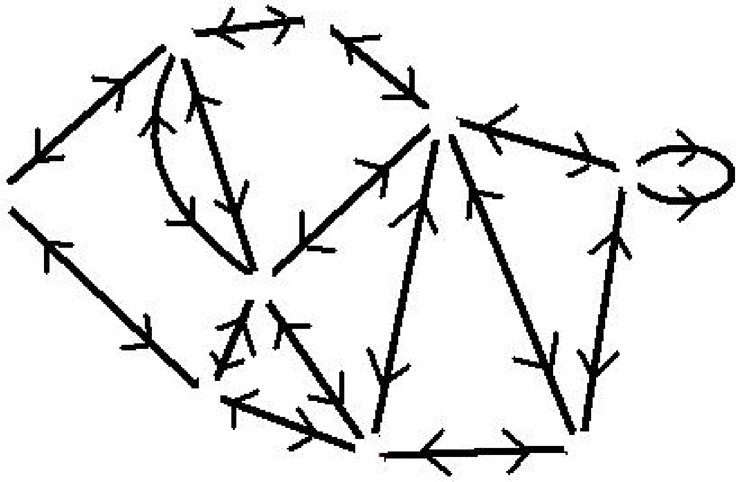


V



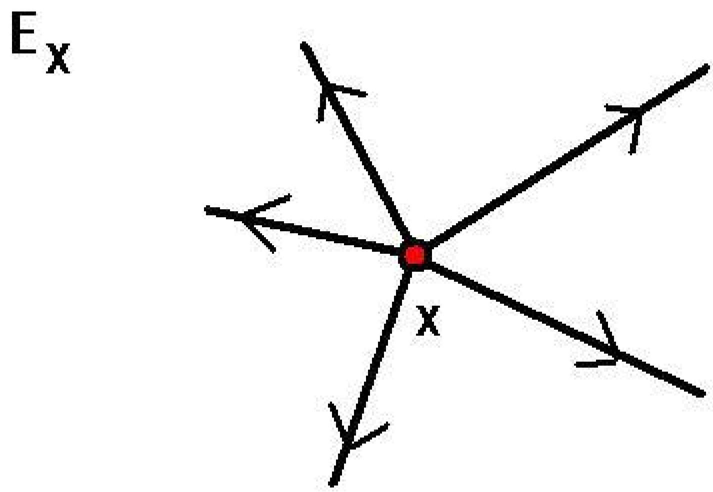
The set of vertices

E



The set of all oriented edges

$\boldsymbol{o(e)}$ **= the origin of** $\boldsymbol{e}$**,** $\boldsymbol{t(e)}$ **= the terminus of** $\boldsymbol{e}$**.**



$|E_x| = \deg x$, **the degree of the vertex** x

- $|E_x|$ **is called the degree of** x**, and written as** $\deg x$**.**

∘ **If deg $x \equiv k$, the graph X is said to be a regular graph of degree k.**

Convention : we write $q+1$ for the degree of a regular graph.

∘ **A subset $E^o \subset E$ is said to be an orientation if $E^o \cup \overline{E^o} = E$, $E^o \cap \overline{E^o} = \emptyset$.**

Graph-theoretic definition of covering graphs

Let $X = (V, E)$, and $X_0 = (V_0, E_0)$. A pair of maps (φ, ψ) is said to be a covering map if

1. $\varphi : V \longrightarrow V_0$ and $\psi : E \longrightarrow E_0$ are surjective,

2. $o\big(\psi(e)\big) = \varphi\big(o(e)\big)$, $t\big(\psi(e)\big) = \varphi\big(t(e)\big)$,

3. $\psi(\overline{e}) = \overline{\psi(e)}$,

4. for every $x \in V$, the restriction $\psi : E_x \longrightarrow E_{0,\varphi(x)}$ is a bijection.

Cayley (Serre) graphs

G : a group,

$i : A \longrightarrow G$: a map of a finite set A into G such that $i(A)$ generates G. We put $q = 2|A| - 1$.

$\overline{A} = \{\overline{a};\ a \in A\}$: a disjoint copy of A.

A word with letters in A means either void (denoted by $\emptyset$) or a finite sequence $w = (b_1, \ldots, b_n)$ with $b_i \in A \cup \overline{A}$.

The length n of a word $w = (b_1, \ldots, b_n)$ is denoted by $|w|$ ($|\emptyset| = 0$).

A word $w = (b_1, \ldots, b_n)$ is said to be reduced if $\overline{b_{i+1}} \neq b_i$ $(i = 1, \ldots, n-1)$, where $\overline{\overline{a}}$ is understood to be a. Denote by $g(w) \in G$ the product $i(b_1) \cdots i(b_n)$ $(g(\emptyset) = 1)$, where $i(\overline{a})$ is understood to be $i(a)^{-1}$.

Given a (G, i, A), the Cayley graph $X = X(G, A)$ is constructed in the following way.

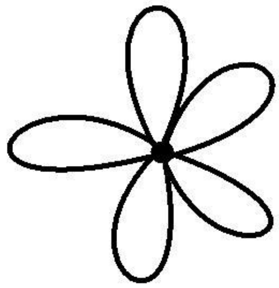
$$V = G, \quad E^o = G \times A,$$
$$o(g, a) = g, \quad t(g, a) = ga$$

Forgetting orientation, we get a connected regular graph $X(G, A)$ of degree $q + 1$

Remark The definiton above of Cayley graphs is slightly different from the conventional one.

The reason why we take up this definition is that, when we consider a group G defined by generators A and relations R ($G = \langle A|R\rangle$), the map of A into G is not necessarily one-to-one (for instance, $G = \langle x, y|x^{-1}yx = y^2,\ y^{-1}zy = z^2,\ z^{-1}xz = z^2\rangle$ is trivial).

∘ $X(G, A)$ is a regular covering graph over a bouquet graph with the covering transformation group G.

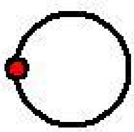


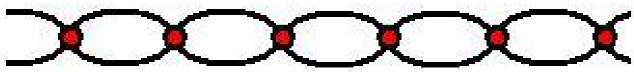
∘ Conversely, A regular covering graph over a bouquet graph is a Cayley graph.

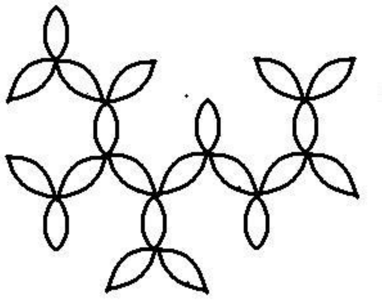
∘ $X(G, A)$ is a tree if and only if G is a free group with the basis A.

∘ **If $F(A)$ is the free group with basis A, then the canonical homomorphism $F(A) \longrightarrow G$ induces the universal covering map $X(F(A), A) \longrightarrow X(G, A)$.**

Example $\mathbb{Z}_2 = \langle a |\ a^2 = 1 \rangle$, $\mathbb{Z}_2 * \mathbb{Z}_2 = \langle a, b |\ a^2 = 1, b^2 = 1 \rangle$







$\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 = \langle a, b, c |\ a^2 = b^2 = c^2 = 1 \rangle$

Homology groups

Let A be an abelian group (for insatnce, $A = \mathbb{Z}, \mathbb{R}$).

The the group of 0-chains

$$C_0(X, A) = \{\sum_x a_x x;\ a_x \in A\}$$

The the group of 1-chains

$$C_1(X, A) = \{\sum_e a_e e;\ a_e \in A\}/\langle e + \overline{e}\rangle,$$

that is, $C_1(X, A)$ is an A-module generated by E with the relation $\overline{e} = -e$.

The boundary map $\partial : C_1(X, A) \longrightarrow C_0(X, A)$ is defined by

$$\partial e = t(e) - o(e)$$

The homology groups are defined as

$$H_0(X, A) = C_0(X, A)/\text{Image } \partial,$$
$$H_1(X, A) = \text{Ker } \partial \ (\subset C_1(X, A))$$

∘ The Euler number

$$\chi(X) = \dim H_0(X, \mathbb{R}) - \dim H_1(X, \mathbb{R})$$
$$= \#V - \#E/2$$

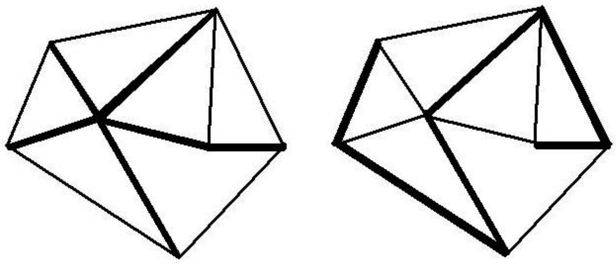
∘ A closed path $c = (e_1, \ldots, e_n)$ gives rise to the homology class $e_1 + \cdots + e_n \in H_1(X, \mathbb{Z})$.

∘ Each $\alpha \in H_1(X, \mathbb{Z})$ is represented by a closed path.

∘ $H_1(X, \mathbb{Z})$ is a lattice (group) of $H_1(X, \mathbb{R})$

The rank of $H_1(X, \mathbb{Z})$ is easily calculated by taking a spanning tree.

Spanning trees



A spanning tree T is a subtree of X containg all vertices of X.

By contracting a spanning tree T to a point, one gets a bouquet graph with n loop edges, where n is the number of non-oriented edges not in T.

X has the same homotopy type with the bouquet graph. Therefore the number of unoriented edges not in T is equal to dim $H_1(X, \mathbb{R})$.

Cohomology groups

Define the groups of 0-cochains and 1-cochains by

$$C^0(X,\mathbb{R}) = \{f : V \longrightarrow \mathbb{R}\}$$
$$C^1(X.\mathbb{R}) = \{\omega : E \longrightarrow \mathbb{R};\ \omega(\overline{e}) = -\omega(e)\}.$$

The coboundary operator $d : C^0(X,\mathbb{R}) \longrightarrow C^1(X,\mathbb{R})$ is defined by

$$df(e) = f(t(e)) - f(o(e)).$$

The cohomology groups are defined as

$$H^0(X,\mathbb{R}) = \text{Ker } d (= \mathbb{R}),$$
$$H^1(X,\mathbb{R}) = C^1(X,\mathbb{R})/\text{Image } d$$

$H^i(X,\mathbb{R})$ is the dual space of $H_i(X,\mathbb{R})$.

Laplacians

By Laplacian, we mean the Laplace-Beltrami operator on a Riemannian manifold or a discrete Laplacian on a graph.

∘ Laplacian on a Riemannian manifold

$$\Delta = \delta d = -\frac{1}{\sqrt{\det g}} \sum_{i,j} \frac{\partial}{\partial x_i} \sqrt{\det g} g^{ij} \frac{\partial}{\partial x_j}.$$

This is a positive operator acting on L^2-functions.

Discrete Laplacians on weighted graphs

$$\Delta = \delta d$$

where $\delta : C^1(X, \mathbb{R}) \longrightarrow C^0(X, \mathbb{R})$ is the (formal) ajoint of d with respect to the inner products on $C^0(X, \mathbb{R})$ and $C^1(X, \mathbb{R})$ defined respectively by

$$\langle f_1, f_2 \rangle = \sum_{x \in V} f_1(x) f_2(x) m_V(x),$$

$$\langle \omega_1, \omega_2 \rangle = \frac{1}{2} \sum_{e \in E} \omega_1(x) \omega_2(x) m_E(e) \quad (m_E(\overline{e}) = m_E(e))$$

Explicitly

$$(\delta\omega)(x) = -\frac{1}{m_V(x)} \sum_{e \in E_x} m_E(e) \omega(e)$$

$$(\Delta f)(x) = -\frac{1}{m_V(x)} \sum_{e \in E_x} m_E(e)\big(f(te) - f(oe)\big)$$

(1) Combinatorial Laplacian (the case $m_V = m_E \equiv 1$)

$$\Delta = \mathcal{D} - \mathcal{A}$$

where $\mathcal{A}$ is the adjacency operator defined as

$$(\mathcal{A} f)(x) = \sum_{e \in E_x} f(t(e))$$

and $\mathcal{D}$ is defined as

$$(\mathcal{D} f)(x) = (\deg x) f(x).$$

(2) Canonical Laplacian (the case $m_V(x) = \deg x,\ m_E \equiv 1$)

$$\Delta = I - \mathfrak{D}^{-1}\mathcal{A}$$

Remark (1) The combinatorial Laplacian appears often in algebraic graph theory.

(2) The canonical Laplacian is related to simple random walks. In fact $\mathfrak{D}^{-1}\mathcal{A}$ is the transition operator for the simple random walk.

From now on, we consider Laplacians acting on L^2-functions.

The Laplacian on a manifold is not bounded, but the canonical Laplacians on graphs is always bounded (the combinatorial Lalacian is bounded if the graph has bounded degree).

3. Twisted Laplacians

Twisted Laplacians

∘ Let $X \xrightarrow{G} X_0$ be a regular covering map over a closed Riemannian manifold or (weighted) finite graph.

Given a unitary representation $\rho : G \to U(W)$, define the Hilbert space ℓ^2_ρ by

$$\ell^2_\rho = \{f : V \to W;\ f(gx) = \rho(g)f(x)\}$$

The inner product is, for instance in the case of canonical Laplacians,

$$\langle f_1, f_2 \rangle = \sum_{x \in \mathcal{F}} \langle f_1(x), f_2(x) \rangle_W (\deg x)$$

where $\mathcal{F}$ is a fundamental set in V for the G-action.

Extend Δ to an operator acting in functions with values in W, and put $\Delta_\rho = \Delta|\ell^2_\rho$ (the twisted Laplacian).

In the case of manifolds, Δ_ρ is the Laplacian acting on sections of the flat vector bundle (possibly of infinite rank) associated with the representation ρ.

Application to isospectral problem

Lemma Let $\rho_H : G \longrightarrow U\big(\ell^2(H\backslash G)\big)$ be the regular representation associated with a subgroup H of G. Then Δ_{ρ_H} is unitarily equivalent to $\Delta_{H\backslash X}$ on the quotient space $H\backslash X$. In particular, Δ_{ρ_1} is unitarily equivalent to Δ_X (an analogue of Theorem of normal basis).

$$\ell^2(H\backslash G) = \{f \in C(H\backslash G);\ \sum_{Hg \in H\backslash G} |f(Hg)|^2 < \infty\}$$

ρ_H is defined as

$$(\rho_H(g)f)(Hg') = f(Hg'g) \quad (f \in \ell^2(H\backslash G)).$$

Theorem of normal basis says that if K is a finite Galois extension of k with Galois group G, then the k-linear representation of G on K is equivalent to the regular representation, or equivalently K is isomorphic to $k[G]$ as a k-linear space.

Let $\mathfrak{F}_G \subset V$ be a fundamental set for the G-action on V. The Hilbert space $\ell^2_{\rho_H}$ is identified with the space of functions $f : V \times (H\backslash G) \longrightarrow \mathrm{C}$ satisfying

$$\begin{aligned} f(gx, Hg') &= f(x, Hg'g) \qquad (g, g' \in G, x \in V), \\ \|f\|^2 &:= \sum_{x \in \mathfrak{F}_G} \sum_{Hg \in H\backslash G} |f(x, Hg)|^2 m(x) < \infty \end{aligned}$$

($m(x) = \deg\, x$). Given $\varphi \in \ell^2(V_{H\backslash X})$, define $f = T(\varphi) \in \ell^2_{\rho_H}$ by setting $f(x, Hg) = \varphi\big(\pi_H(gx)\big)$ where $\pi_H : V \longrightarrow V_{H\backslash X}$ is the canonical map. Since

$$\mathfrak{F}_H := \bigcup_{Hg \in H\backslash G} g\mathfrak{F}_G$$

is a fundamental set for the H-action on V, we have

$$\|f\|^2 = \sum_{x \in \mathfrak{F}_H} |\varphi\big(\pi_H(x)\big)|^2 m(x) = \sum_{x \in V_{H\backslash X}} |\varphi(x)|^2 m(x) = \|\varphi\|^2.$$

It is straightforward to check that T is isometry and commutes with the canonical discrete Laplacians.

Corollary If ρ_{H_1} and ρ_{H_2} are unitarily equivalent for two subgroups H_1, H_2, then $\Delta_{H_1\backslash X}$ and $\Delta_{H_1\backslash X}$ are unitary equivalent.

If G is finite group, then ρ_{H_1} and ρ_{H_2} are equivalent if and only if $|[g] \cap H_1| = |[g] \cap H_2|$ for every $g \in G$. The corollary provides us a method to construct isospectral manifolds and graphs.

The corollary above is an analogue of the following:

Let K be a finite Galois extension of Q with Galois group $G = G(K/\mathrm{Q})$, and let k_1 and k_2 be subfields of K corresponding to subgroups H_1 and H_2, respectively. Then the following two conditions are equivalent:

(1) Each conjugacy class of elements in G meets H_1 and H_2 in the same number of elements.

(2) The Dedekind zeta functions of k_1 and k_2 are the same.

Digression

Theorem Let G be a finite group. Under the condition in the corollary above, the manifolds (finite graphs) $H_1\backslash X$ and $H_2\backslash X$ are iso-length spectral, in the sense that for each $x \geq 0$, there is a 1-to-1 correspondence between the sets

$$\{\mathfrak{p}_1;\ \text{prime geodesic cycles in } H_1\backslash X \text{ with } \ell(\mathfrak{p}_1) = x\}$$

and

$$\{\mathfrak{p}_2;\ \text{prime geodesic cycles in } H_2\backslash X \text{ with } \ell(\mathfrak{p}_2) = x\}$$

The proof relies on the fact that one can establish an analogue of algebraic number theory in which prime geodesic cycles play a similar role as prime ideals in number fields.

The (geometric) zeta function $Z_X(s)$ of a closed manifold (finite graph) X is defined as

$$Z_X(s) = \prod_{\mathfrak{p}} \left(1 - e^{-s\ell(\mathfrak{p})}\right)^{-1}$$

$$(Z_X(u) = \prod_{\mathfrak{p} \in P} (1 - u^{|\mathfrak{p}|})^{-1})$$

This is an analogue of Dedekind zeta functions for number fields.

Corollary Under the same condition as in the theorem above, $Z_{H_1\backslash X}(s) = Z_{H_2\backslash X}(s)$ $(Z_{H_1\backslash X}(u) = Z_{H_2\backslash X}(u))$.

Kazhdan distance

Let G be an arbitrary discrete group.

∘ Let $\rho : G \longrightarrow U(W)$ be a unitary representation on a Hilbert space W. Define $\delta(\rho, 1)$, the "distance" between the trivial representation 1 and ρ, by

$$\delta(\rho, 1) = \inf_{\substack{v \in W \\ \|v\|=1}} \sup_{g \in A} \|\rho(g)v - v\|$$

where A is a finite set of generators.

> **Theorem** Let $\lambda_0(\rho) = \inf \sigma(\Delta_\rho)$. There exist positive constants c_1, c_2 not depending on ρ such that
>
> $$c_1 \delta(\rho, 1)^2 \leq \lambda_0(\rho) \leq c_2 \delta(\rho, 1)^2$$

In particular, $\lambda_0(\rho) = 0$ if and only if $\delta(\rho, 1) = 0$.

Corollary(R. Brooks) $\lambda_0(\Delta_X) = 0$ if and only if G is amenable.

This is a consequence of the fact that $\delta(\rho_1, 1) = 0$ if and only if G is amenable.

Amenable groups

A discrete group G is said to be amenable if it has a (left) invariant mean; that is, a continuous linear functional m on the Banach space $\ell^\infty(G, \mathbb{R})$ satisfying the following properties :

(1) $m(1) = 1$,

(2) if $f \geq 0$ and $f \in \ell^\infty(G, \mathbb{R})$, then $m(f) \geq 0$, and

(3) $m(\sigma f) = m(f)$, where $(\sigma f)(\mu) = f(\sigma^{-1}\mu)$ $(\sigma \in G,\ f \in \ell^\infty(G, \mathbb{R}))$.

Idea of the proof is to use the expression of $\boldsymbol{\lambda}_0(\boldsymbol{\rho})$:

$$\boldsymbol{\lambda}_0(\boldsymbol{\rho}) = \inf_{f \in \ell^2(\rho)} \frac{\int \|\boldsymbol{d}_\rho \boldsymbol{f}\|^2}{\int \|\boldsymbol{f}\|^2}$$

Kazhdan groups

G is said to have the Kazhdan property (T) (or to be a Kazhdan group) if there exists a positive constant c such that $\delta(\rho, 1) \geq c$ for every non-trivial irreducible representation ρ of G.

A typical example of Kazhdan groups is $SL_n(\mathbb{Z})$ $(n \geq 3)$. The rotation group $SO(n)$ $(n \geq 5)$ has a finitely generated dense Kazhdan subgroup.

Expanders

Let $\cdots \to X_n \to \cdots \to X_1 \to X_0$ be a sequence of finite-fold covering maps. Suppose that every covering map $X_n \longrightarrow X_0$ is a subcovering map of a fixed regular covering map $X \xrightarrow{G} X_0$.

> **Theorem** **If G is a Kazhdan group, then $\{X_n\}$ is a family of expanders, namely there exists a positive constant c such that $\lambda_1(X_n) \geq c$ for every n.**

A family of expanders is a model of efficient communication networks.

Ruziewicz's problem

This asks the uniqueness of rotationally invariant finitely additive measures defined on Lebesgue sets on S^{n-1}

The proof of uniqueness reduces to the existence of an ϵ-good set in $SO(n)$, where a finite set $A \subset SO(n)$ said to be an ϵ-good set if $\|L_a f - f\|_2 \geq \epsilon \|f\|_2$ for $a \in A$ and $f \in L^2(S^{n-1})$ with $\displaystyle\int_{S^{n-1}} f = 0$.

If the group G generated by A is dense in $SO(n)$ and has the property (T), then A is an ϵ-good set for some $\epsilon > 0$.

You just make use of the representation of G on $L^2(S^{n-1})$, and consider the twisted Laplacian on the Cayley graph $X(G, A)$.

Spectra of abel covers

Let $X \xrightarrow{G} X_0$ be an abelian covering map (G is supposed to be an infinite abelian group).

The regular representation ρ_1 of the abelian group G is decomposed as

$$\rho_1 = \int_{\widehat{G}}^{\oplus} \chi \, d\chi,$$

where $\widehat{G}$ is the group of unitary characters (homomorphism of G into $U(1)$) with the normalized Haar measure $d\chi$.

Associated with this irreducible decomposition is the following direct integral decomposition:

$$\Delta_X = \int_{\widehat{G}}^{\oplus} \Delta_\chi d\chi$$

Let

$$0 \le \lambda_0(\chi) \le \lambda_1(\chi) \le \cdots \le \lambda_{N-1}(\chi)$$

be the eigenvalues of Δ_χ. Each λ_i is a continuous function on $\widehat{G}$.

Theorem $\sigma(\Delta_X) = \bigcup_{i=0}^{N-1}\{\lambda_i(\chi);\ \chi \in \widehat{G}\}$ **(in the case of manifolds, N is understood to be ∞).**

- Any $\chi \in \widehat{G}$ is written as

$$\chi(g) = \exp\left(2\pi\sqrt{-1}\int_{C_g}\omega\right)$$

with a harminic 1-form ω ($\delta\omega = 0$)), where C_g is a closed path such that $\mu(C_g) = g$ ($\mu : H_1(X_0, \mathbb{Z}) \to G$ is the homomorphism associated with the covering map).

- In the case of graphs, define $\Delta_\omega : C(V_0) \to C(V_0)$ by

$$(\Delta_\omega f)(x) = \frac{1}{\deg x}\Big(\sum_{e \in E_x} e^{2\pi\sqrt{-1}\omega(e)} f(te)\Big) - f(x)$$

$(\Delta_\chi, \ell^2_\chi)$ is unitarily equivalent to $(\Delta_\omega, \ell^2(V_0))$.

No-gap conjecture (1) For the maximal abelian covering graph X over a finite regular graph, $\sigma(-\Delta_X) = [0, 2]$.

(2) For the maximal abelian covering surface X over a closed surface with constant negative curvature, $\sigma(-\Delta_X) = [0, \infty)$.

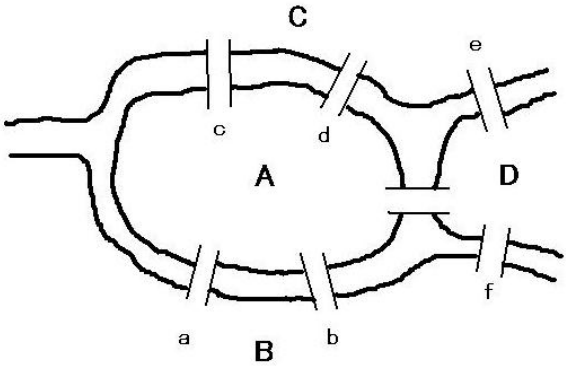
Theorem (Yu.Higuchi) Let $X \xrightarrow{G} X_0$ be the maximal abelian covering graph of arbitrary finite graph. If deg x is even for every vertex $x \in X_0$, then $\sigma(-\Delta_X) = [0, 2]$.

Proof From the assumption, we have a closed path (Euler path) c in X_0 such that every unoriented edge occurs in $c = (e_1, \ldots, e_n)$ once and only once (the famous solution to "the puzzle of the seven bridges" due to Euler).

∘ Define ω by setting $\omega(e_i) = 1, \omega(\overline{e}_i) = -1$, otherwise $\omega(e) = 0$. ω is a harmonic 1-form ($\sum_{e \in E_{0x}} \omega(e) = 0$). With this ω,

$$\sum_{e \in E_{0x}} \exp(2\pi\sqrt{-1}t\omega) = (\deg x) \cos 2\pi t a,$$

so that $\Delta_{t\omega} 1 = (\cos 2\pi t a - 1)1$. From this observation, we conclude $\sigma(-\Delta_X) = [0, 2]$.



4. Analysis on Cayley graphs

Cogrowth and spectra of finitely generated groups

The cogrowth sequence $\{\ell_n\}_{n=0}^{\infty}$ of (G, A) is defined by

$$\ell_n = \big|\{w;\ w \text{ is a reduced word over } A \text{ with } g(w) = 1 \text{ and } |w| \le n\}\big|.$$

Remember that, for a word $w = (b_1, \dots, b_n)$, the notation $g(w) \in G$ means the product $i(b_1) \cdots i(b_n)$ $(g(\emptyset) = 1)$, where $i(\overline{a})$ is understood to be $i(a)^{-1}$.

Thus $\{\ell_n\}_{n=0}^{\infty}$ is a counting function for relations.

Theorem (**Grigorchuk**)

(i) $\ell = \lim_{n\to\infty} \ell_n^{1/n}$ exists.

(ii) $1 \le \ell \le q$;

(iii) $\ell = 1$ if and only if G is the free group with the basis A;

(iv) $\ell = q$ if and only if G is amenable;

(v) if G is not a free group, then $q^{1/2} < \ell \le q$.

Recall that $q + 1 = 2|A|$.

The adjacency operator on the Cayley graph $X(G, A)$ is expressed as

$$\mathcal{A}f(g) = \sum_{a \in A} \big[f\big(g \cdot i(a)\big) + f\big(g \cdot i(a)^{-1}\big) \big].$$

Note $\mathcal{A} : \ell^2(G) \longrightarrow \ell^2(G)$ is G-equivariant.

The cogrowth sequence is directly related to the adjacency operator $\mathcal{A}$ by the formula

$$\sum_{n=0}^{\infty} \ell_n z^n = \mathrm{tr}_G \left(\frac{1+z}{1 - \mathcal{A}z + qz^2} \right).$$

Here $\mathrm{tr}_G T = \langle T\delta_1, \delta_1 \rangle$ for a G-equivariant operator T.

If G is finite, then

$$\mathrm{tr}_G T = \frac{1}{|G|} \mathrm{tr}\ T$$

Theorem Put $\alpha = \sup \sigma(\mathcal{A})$. Then

(i) $2q^{1/2} \leq \alpha \leq q + 1$;

(ii) $\alpha = 2q^{1/2}$ if and only if G is a free group with the basis A;

(iii) $\alpha = q + 1$ if and only if G is amenable;

(iv) $\ell = \big(\alpha + (\alpha^2 - 4q)^{1/2}\big)/2$ provided that G is not free.

The cogrowth rate is a complementary concept of the growth rate which is defined as

$$\boldsymbol{b = \lim_{n\to\infty} b_n^{1/n}}$$

where

$$\boldsymbol{b_n = \left|\{g \in G;\ \text{there exists a word } w \text{ with } g = g(w),\ |w| \le n\}\right|.}$$

Theorem **(i)** $\boldsymbol{1 \le b \le q}$**;**

(ii) if $\boldsymbol{b = 1}$**, then** $\boldsymbol{G}$ **is amenable;**

(iii) if $\boldsymbol{G}$ **is the free group with the basis** $\boldsymbol{A}$**, then** $\boldsymbol{b = q}$**.**

(iv) (K. Fujiwara)

$$\boldsymbol{\sup \sigma(\mathcal{A}) \ \ge \frac{2(q+1)}{b^{1/2} + b^{-1/2}}.}$$

Zeta functions of finitely generated groups

A word $w = (b_1, \dots, b_n)$ is said to be cyclically reduced if $\overline{b_{i+1}} \neq b_i$ $(i = 1, 2, \dots, n-1)$ and $\overline{b_1} \neq b_n$.

A cyclically reduced word w is said to be prime if it is not a power of another word.

Two words w_1 and w_2 are equivalent if w_1 is obtained from w_2 by a cyclic permutation.

Let P be the set of equivalence classes of cyclically reduced prime words w with $g(w) = 1$.

Define the zeta function $Z(u)$ by

$$Z(u) = \prod_{\mathfrak{p} \in P} (1 - u^{|\mathfrak{p}|})^{-1}.$$

$$Z(u) = (1-u^2)^{-(q-1)/2}\det_G(1-\mathcal{A}u+qu^2)^{-1},$$

where $\det_G$ stands for the G-determinant defined by $\det_G(T) = \exp \operatorname{tr}_G(\log T)$

What can one say about analytic properties of $Z(u)$?

Counting "lattice points"

$F = F(A)$: the free group with the free basis A. Let $H = \mathrm{Ker}(F \longrightarrow G)$.

$X(G, A) = H \backslash X(F, A)$.

(Recall $X(F, A)$ is the universal covering graph over $X(G, A)$),

Applying the path-lifting property of covering maps, we have

$$\ell_n = \big|\{h \in H;\ d(1, h) \leq n\}\big|,$$

where d is the distance function on $X(F, H)$.

Put

$$t_n = \sum_{k=0}^{[n/2]} \big|\{h \in H;\ d(1, h) = n - 2k\}\big|.$$

$$\sum_{n=0}^{\infty} t_n z^n = \mathrm{tr}_G \frac{1}{1 - \mathcal{A} z + q z^2}.$$

Recall that the Chebychev polynomial U_n of the second kind is defined by $U_n(\cos\theta) = \sin(n+1)\theta / \sin\theta$ and satisfies

$$\sum_{n=0}^{\infty} U_n(\mu) z^n = \frac{1}{1 - 2\mu z + z^2}.$$

Thus

$$t_n = q^{n/2} \mathrm{tr}_G U_n\Big(\frac{1}{2\sqrt{q}} \mathcal{A}\Big).$$

If G is finite,

$$t_n = \frac{q^{n/2}}{N} \sum_{i=0}^{N-1} U_n\Big(\frac{\mu_i}{2\sqrt{q}}\Big),$$

where $N = |G|$ and $q+1 = \mu_0 > \mu_1 \geq \cdots \geq \mu_{N-1} \geq -(q+1)$ are eigenvalues of $\mathcal{A}$.

Remark $q+1$ is the maximal eigenvalue. $-(q+1)$ is eigenvalue if and only if the graph is bipartite.

$$\begin{cases} q^{n/2} U_n\Big(\frac{q+1}{2\sqrt{q}}\Big) = \frac{q^{n+1}-1}{q-1} = \sum_{d|q^n} d, \\ q^{n/2} U_n\Big(\frac{-(q+1)}{2\sqrt{q}}\Big) = (-1)^n \frac{q^{n+1}-1}{q-1} = (-1)^n \sum_{d|q^n} d, \\ q^{n/2} U_n\Big(\frac{\mu_i}{2\sqrt{q}}\Big) = o(q^n) \quad (|\mu_i| < q+1). \end{cases}$$

Ramanujan graphs

A finite regular graph of degree $q+1$ is said to be a Ramanujan graph if every eigenvalues μ_i of $\mathcal{A}$ except for $\pm(q+1)$ satisfies $|\mu_i| \leq 2\sqrt{q}$.

Remark (1) A graph is Ramanujan if and only if the zeta function satisfies the Riemannian Hypothesis.

(2) A family of Ramanujan graphs is the "best" family of expanders.

When $H = F$ (G is trivial), we have $t_n = \sum_{d|q^n} d$, and $8t_n$ coincides with the number of representations of q^n as a sum of 4 squares provided that q is an odd prime (Jacobi).

Problem Find a criterion for a normal subgroup H of finite index in $F(A)$ such that an appropriate multiple of t_n is the number of representations of q^n by an integral quadratic form (of 4 variables).

Example (Lubotzky, Phillips and Sarnak) There exists $H = H(p)$ such that $2t_n$ is expressed as the number of representatives of q^n by the quadratic form ${x_1}^2 + (2p)^2{x_2}^2 + (2p)^2{x_3}^2 + (2p)^2{x_4}^2$, where p, q are unequal primes both $\equiv 1 \pmod 4$.

$\Longrightarrow$ **$2t_n$ is the Fourier coefficient of a modular form of weight two for the congruence subgroup $\Gamma(16p^2)$**

$\Longrightarrow$ **$2t_n$ is expressed as the sum of the Fourier coefficient $a(q^n)$ of a cusp form and the coefficient $\delta(q^n)$ of an Eisenstein series.**

Fact (1) $\delta(q^n) = \sum_{d|q^n} dS(d)$ **with a periodic function S on $\mathbb{N}$.**
(2) **Ramanujan conjecture** (now a theorem)
$a(q^n) = O_\epsilon(q^{n(1/2+\epsilon)})$ **for an arbitrary positive ϵ.**

Fact **If $\sum_{d|q^n} dR(d) = o(q^n)$ for a periodic function R on $\mathbb{N}$, then $\sum_{d|q^n} dR(d) = 0$.**

$$t_n = \frac{q^{n/2}}{N} \sum_{i=0}^{N-1} U_n\Big(\frac{\mu_i}{2\sqrt{q}}\Big),$$

$$N(a(q^n)+\delta(q^n)) = 2\sum_{d|q^n} d + 2(-1)^n \sum_{d|q^n} d + 2q^{n/2} \sum_{i=1}^{N-2} U_n\Big(\frac{\mu_i}{2\sqrt{q}}\Big)$$

$$\begin{cases} q^{n/2} U_n\Big(\frac{q+1}{2\sqrt{q}}\Big) = \frac{q^{n+1}-1}{q-1} = \sum_{d|q^n} d, \\ q^{n/2} U_n\Big(\frac{-(q+1)}{2\sqrt{q}}\Big) = (-1)^n \frac{q^{n+1}-1}{q-1} = (-1)^n \sum_{d|q^n} d, \\ q^{n/2} U_n\Big(\frac{\mu_i}{2\sqrt{q}}\Big) = o(q^n) \quad (|\mu_i| < q+1). \end{cases}$$

$$\Longrightarrow \;\; N\delta(q^n) - 2\sum_{d|q^n} d - 2(-1)^n \sum_{d|q^n} d = o(q^n)$$
$$\Longrightarrow \;\; N\delta(q^n) - 2\sum_{d|q^n} d - 2(-1)^n \sum_{d|q^n} d = 0$$
$$\Longrightarrow \;\; \sum_{i=1}^{N-2} U_n\Big(\frac{\mu_i}{2\sqrt{q}}\Big) = O_\epsilon(q^{n\epsilon})$$
$$\Longrightarrow \;\; |\mu_i| \le 2\sqrt{q} \text{ for } 1 \le i \le N-1$$