Discrete Abel-Jacobi Maps and Albanses maps in graph theory

Toshikazu Sunada

Department of Mathematics, Meiji University

The purpose of this talk

To give a relationship between the notion of discrete Abel-Jacobi maps, which has been introduced in algebraic graph theory, and the notion of Albanese maps in graph theory, which is related to random walks on crystal lattices.

For an illustration, we look at two examples of Albanese maps, which have something to do with the diamond crystal and its twin.

Background

Abel-Jacobi maps, Albanese maps in algebraic geometry $\Longleftarrow$ The studies of algebraic functions by Gauss, Abel, Jacobi, Riemann, and others in 19th century.

Gauss, Abel, and Jacobi studied the (complex) integral of the form

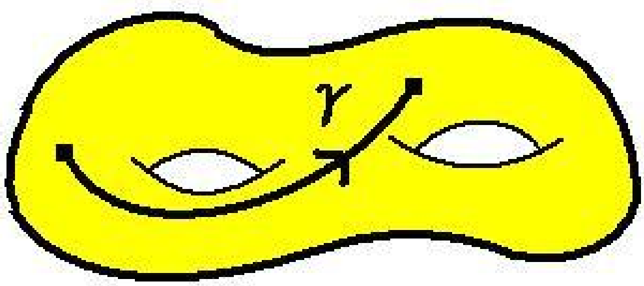
$$\int_\gamma R(z, \sqrt{\varphi(z)})dz,$$

where $R(z, w)$ is a rational function, and $\varphi(z)$ is a polynomial of degree 3 or 4. This is what you call an elliptic integrals.

Riemann introduced what we call "Riemann surfaces" today, with which the integral above is written as

$$\int_\gamma \omega,$$

where ω is a meromorphic 1-form on the Riemann surface (algebraic curve) defined by $w^2 = \varphi(z)$. This view led to the study of integrals over more general algebraic curves.



Notations and conventions

- A graph is denoted as $X = (V, E)$, where

$$V = \text{the set of vertices},$$
$$E = \text{the set of all oriented edges}$$

$$o(e) = \text{the origin of } e \in E,$$
$$t(e) = \text{the terminus of } e \in E,$$
$$\bar{e} = \text{the inversion of } e \in E.$$

$$E_x = \{e \in E;\ o(e) = x\}$$

I. Abel's Theorem in Graph Theory

Abel-Jacobi maps in graph theory

Abel-Jacobi maps in the original sense are canonical holomorphic mapping of algebraic varieties into certain abelian varieties.

◦ An Abel-Jacobi map in graph theory is a canonical harmonic function on a finite graph with values in a certain finite abelian group, thus a purely discrete object, introduced by R. Bacher, P. de la Harpe and T. Nagnibeda.

References

[1] R. Bacher, P. de la Harpe and T. Nagnibeda, The lattice of integral flows and the lattice of integral cuts on a finite graph, Bull. Soc. math. France, 125, 1997, p. 167-198.

[2] N. Biggs. Algebraic potential theory on graphs. Bull. London Math. Soc., 29(6):641-682, 1997.

[3] M. Baker and S. Norine, Riemann-Roch and Abel-Jacobi theory on a finite graph, Adv. in Math., 215(2007), 766-788.

Abel-Jacobi maps in algebraic geometry

Given a non-singular complete algebraic curve $V/\mathbb{C}$, we write

$$\mathrm{Div}^0(V) = \{ \sum_{x \in V}^{finite} a_x x | \; a_x \in \mathbb{Z}, \; \sum a_x = 0\},$$

and let $P(V)$ be the subgroup of $\mathrm{Div}^0(V)$ consisting of divisors associated with meromorphic functions on V. say

$$\mathrm{Prin}(X) = \{\sum_x \mathrm{ord}_x(f)x| \; f \text{ meromorphic function}\}$$

Then define the Picard variety

$$\mathrm{Pic}(V) := \mathrm{Div}^0(V)/\mathrm{Prin}(V).$$

The Abel-Jacobi map $\Phi : V \longrightarrow \mathrm{Pic}(V)$ is defined by

$$\Phi(x) = [x - x_0] \in \mathrm{Div}^0(V)/\mathrm{Prin}(V).$$

Abel-Jacobi maps for finite graphs

Let $X = (V, E)$ be a finite graph, where V is the set of vertices, and E is the set of all oriented edges. Define

$$\mathrm{Div}^0(X) = \{\sum_{x \in V} a_x x \in C_0(X, \mathbb{Z}) | \sum_x a_x = 0\}$$

$$\mathrm{Prin}(X) = \partial\partial^*\big(C_0(X, \mathbb{Z})\big)$$

where $\partial : C_1(X, \mathbb{Z}) \longrightarrow C_0(X, \mathbb{Z})$ is the boundary operator of chain groups, and ∂^* is the adjoint of ∂ with respect to the inner products on $C_0(X, \mathbb{R})$ and $C_1(X, \mathbb{R})$

$$x \cdot y = \begin{cases} 1 & (x = y) \\ 0 & (x \neq y) \end{cases} \quad e \cdot e' = \begin{cases} 1 & (e = e') \\ -1 & (e = \overline{e'}) \\ 0 & (otherwise), \end{cases}$$

The Picard group is defined as

$$\mathrm{Pic}(X)) = \mathrm{Div}^0(X)/\mathrm{Prin}(X)$$

Pic(X) is a finite abelian group whose order is $\kappa(X)$, the number of spanning trees of X (the tree number).

The Abel-Jacobi map $\Phi : V \longrightarrow \mathrm{Pic}(X)$ is defined as

$$\Phi(x) = [x - x_0].$$

Usually the tree number is very big. For instance, $\kappa(K_n) = n^{n-2}$ (Cayley, 1839). Therefore the Picard group is very big finite abelian group.

Why putting $\mathrm{Prin}(X) = \partial\partial^*(C_0(X,\mathbb{Z}))$?

In the case of algebraic curves,

$$\Delta \log |f| = 2\pi \sum \mathrm{ord}_p(f)\delta_p$$

for a meromorphic function f.

If we identify $C_0(X,\mathbb{R})$ wit $C^0(X,\mathbb{R})$, the group of 0-cochains (= the space of functions on the set of vertices), then

$$\mathrm{Prin}(X) = \Delta\big(C^0(X,\mathbb{Z})\big),$$

where Δ is the combinatorial Laplacian defined as

$$\begin{aligned}\Delta f(x) &= d^* df(x) = -\sum_{e\in E_x} f(te) + (\deg x)f(x) \\ &= -\sum_{e\in E_x} \big(f(te) - f(oe)\big).\end{aligned}$$

$\boldsymbol{\Phi}$ **is a harmonic function in the sense that**

$$\Delta\Phi(x) = \sum_{e \in E_x} [\Phi(t(e)) - \Phi(o(e))] = 0$$

Abel-Jacobi maps have the universal property: If $\varphi : V \longrightarrow G$ be a harmonic function with values in a finite abelian group G, then there exists a unique homomorphism $f : \mathrm{Pic}(X) \longrightarrow G$ such that $f \circ \Phi = \varphi$.

Albanese maps in graph theory

Albanese maps in the original sense are also canonical holomorphic mapping of algebraic varieties into certain abelian varieties.

○ **An Albanese map in graph thoery is a canonical harmonic map from a finite graph (as a 1-dimensional singular space) into a certain flat torus.**

[4] **M. Kotani and T. Sunada, Standard realizations of crystal lattices via harmonic maps, Trans. A.M.S. 353 (2000), 1-20.**

Albanese maps in algebraic geometry

Given a non-singular complete algebraic curve $V/\mathbb{C}$, we let

$$A(V) = (\Omega^1(V))^*/H_1(V, \mathbb{Z}),$$

where

$$\Omega^1(V) = \text{the space of holomorphic 1-forms on } V,$$

and $H_1(V, \mathbb{Z})$ is considered as the subgroup of $(\Omega^1(V))^*$ by using the paring map

$$([\alpha], \omega) = \int_\alpha \omega.$$

$A(V)$ is a complex torus, and called the Albanese torus.

The Albanese map Φ is a holomorphic map of V into $J(V)$ defined by the paring map

$$(\Phi(x), \omega) = \int_{x_0}^{x} \omega \pmod{H_1(V, \mathbb{Z})},$$

where

$$x_0 \in V \text{ is a reference point, and}$$

$$\omega \in \Omega^1(V).$$

The symbol "mod" implies that the linear functional $\omega \mapsto \int_{x_0}^{x} \omega$ in the right hand side is determined only "modulo elements in $H_1(V, \mathbb{Z})$".

Abel's theorem

○ **The correspondence $[x - x_0] \mapsto \Phi(x)$ gives an isomorphism of $\mathrm{Pic}(V)$ onto $A(M)$. Thus under the identification between $\mathrm{Pic}(V)$ and $A(M)$, the Abel-Jacobi map coincides with the Albanese map.**

Albanese maps for finite graphs

Consider the flat torus (Albanese torus)

$$\mathbb{A}(X) = H_1(X, \mathbb{R})/H_1(X, \mathbb{Z}),$$

with the flat metric induced from the inner product on $C_1(X, \mathbb{R})$.

Let $X^{ab} = (V^{ab}, E^{ab})$ be the maximal abelian covering graph; in other words, X^{ab} is the covering graph over X with the covering transformation group $H_1(X, \mathbb{Z})$. Let $P : C_1(X, \mathbb{R}) \longrightarrow H_1(X, \mathbb{R}) = \mathrm{Ker}\ \partial$ be the orthogonal projection. Fix a reference point $x_0 \in V^{ab}$, and let $\widetilde{c} = (\widetilde{e_1}, \dots, \widetilde{e_n})$ be a path in X^{ab} with $o(\widetilde{c}) = x_0, t(\widetilde{c}) = x$. Then put $\widetilde{\Phi}(x_0) = 0$ and

$$\widetilde{\Phi}(x) = P(e_1 + \cdots + e_n) = P(e_1) + \cdots + P(e_n),$$

where $e_i \in E$ is the image of $\widetilde{e_i}$ by the covering map.

- $\widetilde{\Phi} : V^{ab} \longrightarrow H_1(X, \mathbb{R})$ is well-defined.
- Φ is also defined as

$$(\Phi(x), \omega) = \int_{x_0}^{x} \omega \pmod{H_1(X, \mathbb{Z})},$$

where ω is a "harmonic form", that is, $\omega \in C^1(X, \mathbb{R})$ with

$$d^*\omega(x) = -\sum_{e \in E_x} \omega(e) = 0.$$

- The map $\widetilde{\Phi}$ is harmonic in the sense that

$$\Delta\widetilde{\Phi}(x) = -\sum_{e \in E_x} \left[\widetilde{\Phi}(te) - \widetilde{\Phi}(oe)\right] = 0.$$

We extend $\widetilde{\Phi}$ to X^{ab} as a piecewise linear map. Then $\widetilde{\Phi}$ satisfies $\widetilde{\Phi}(\sigma x) = \widetilde{\Phi}(x) + \sigma$ for $\sigma \in H_1(X, \mathbb{Z})$

Therefore $\widetilde{\Phi}$ induces a map $\Phi : X \longrightarrow H_1(X, \mathbb{R})/H_1(X, \mathbb{Z})$, which is a harmonic map of X into the flat torus.

$\Phi : X \longrightarrow \mathbb{A}(X)$ is said to be the Albanese map, and $\widetilde{\Phi}$ will be called the standard realization of X^{ab}

Albanese maps have also the universal property.

What is a relation between Abel-Jacobi maps and Albanese maps ?

1. The homology group $H_1(X, \mathbb{Z})$ is an integral lattice in $H_1(X, \mathbb{R})$ in the sense that $\alpha \cdot \beta \in \mathbb{Z}$ for $\alpha, \beta \in H_1(X, \mathbb{Z})$.

2. Therefore the dual lattice

$$H_1(X, \mathbb{Z})^{\#} = \{\alpha \in H_1(X, \mathbb{R});\ \alpha\cdot\beta \in \mathbb{Z} \text{ for every } \beta \in H_1(X, \mathbb{Z})\}$$

contains $H_1(X, \mathbb{Z})$.

3. The discrete Albanese torus $A(X)$, a finite subgroup of $\mathbb{A}(X)$, is defined to be the quotient group

$$H_1(X, \mathbb{Z})^{\#}/H_1(X, \mathbb{Z}).$$

Analogue of Abel's theorem

Theorem (M. Kotani and Sunada) $\Phi(V) \subset A(X)$, and that $A(X)$ is isomorphic to $\mathrm{Pic}(X)$ in a canonical way. Under the identification $A(X) = \mathrm{Pic}(X)$, the Albanese map as a map of V into $A(X)$ coincides with the Abel-Jacobi map.

How to construct $\widetilde{\Phi}$

1. Take a $\mathbb{Z}$-basis $c_1, \cdots, c_b$ of $H_1(X, \mathbb{Z})$, say consisting of closed paths in X ($b =$ rank $H_1(X, \mathbb{Z})$).
2. Compute $\lambda_{ij} = c_i \cdot c_j$, which gives a complete description of the lattice group $H_1(X, \mathbb{Z})$ in the Euclidean space $H_1(X, \mathbb{R})$. Note λ_{ij} is an integer.
3. For $e \in E$, express $P(e)$ as a linear combination of c_i's:

$$P(e) = \sum_{i=1}^{b} a_i(e) c_i.$$

To obtain $a_i(e)$, compute

$$b_j(e) = P(e) \cdot c_j = e \cdot P(c_j) = e \cdot c_j \in \mathbb{Z}.$$

Then

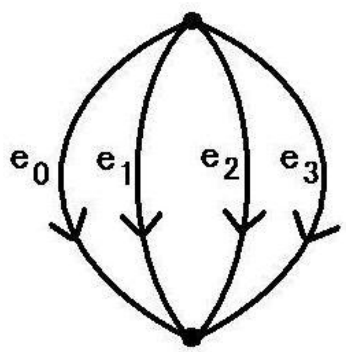
$$\boldsymbol{b_i(e) = \sum_{i=1}^{b} a_i(e)\lambda_{ij}}$$

from which we obtain $a_i(e)$.

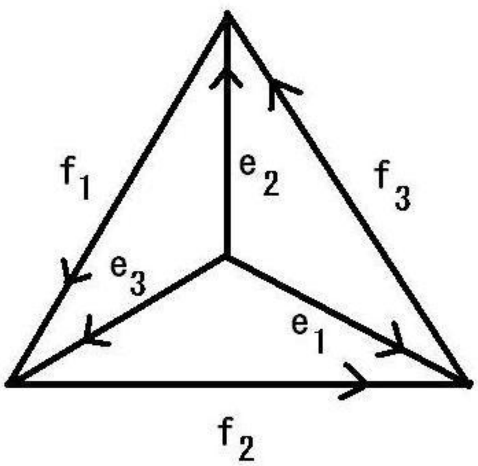
The map $\widetilde{\Phi}$ helps us to understand how the maximal abel cover X^{ab} looks like.

Examples

Consider the graphs (A) and (B). The graph (B) is nothing but the complete graph K_4.

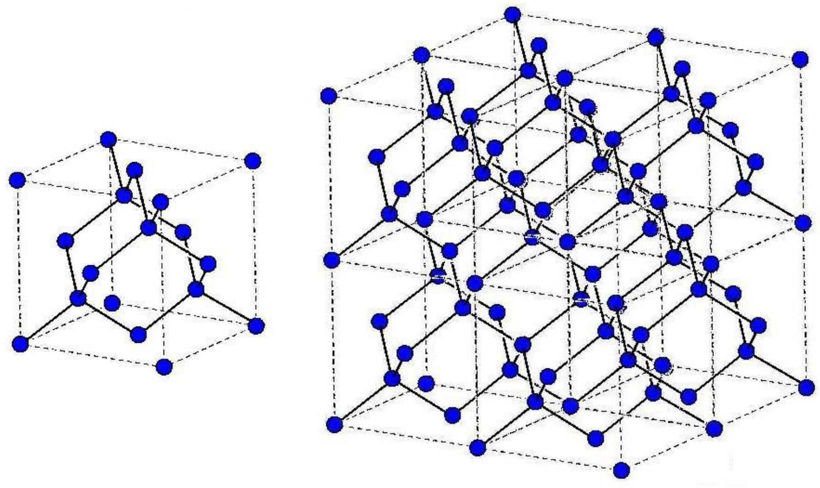


(A)



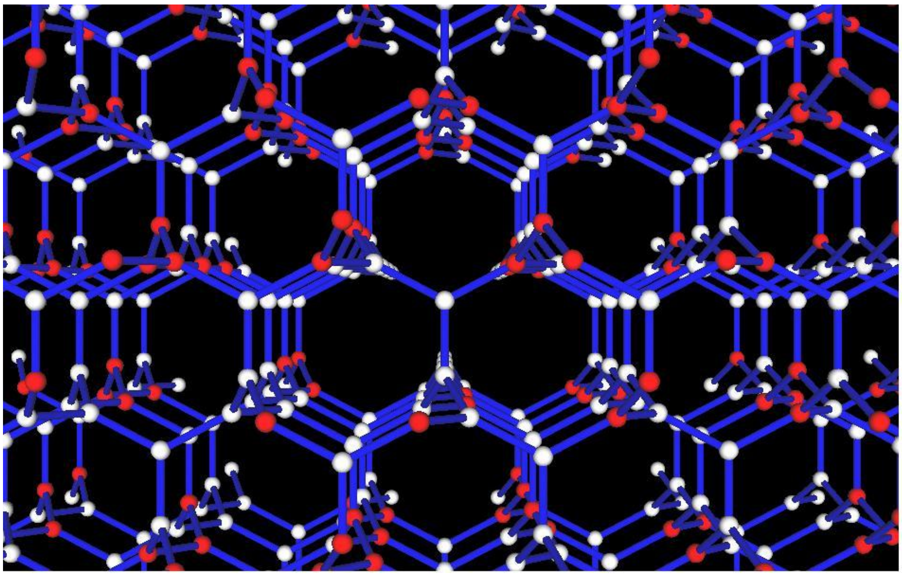
(B)

(A) The image $\widetilde{\boldsymbol{\Phi}}(\boldsymbol{X}^{ab})$ which turns out to be the diamond crystal describes the arrangement of carbon atoms (corresponding to vertices) together with their bondings (corresponding to edges) in the diamond.



Diamond crystal (formed by a web of hexagonal rings)

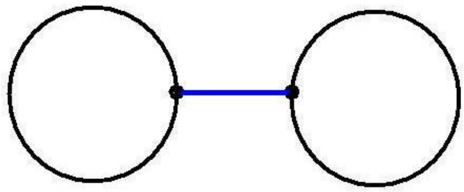
(B) The image $\widetilde{\Phi}(X^{ab})$ is what we call the K_4 crystal.

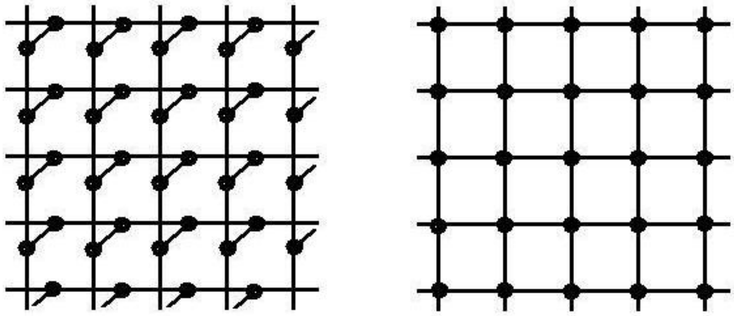


K_4 crystal (formed by a web of decagonal ring)

Properties of $\widetilde{\Phi}$

An edge e degenerates in $H_1(X, \mathbb{R})$ (i.e. $P(e) = 0$) if and only if e is a separating edge (meaning that if we remove e from X, then the resulting graph is not connected).





Suppose that X has no separating edges. Then

(1) $\widetilde{\Phi} : V^{ab} \longrightarrow H_1(X, \mathbb{R})$ is injective.

(2) $\widetilde{\Phi} : E_x^{ab} \longrightarrow H_1(X, \mathbb{R})$ given by

$$e \mapsto P(e)/\|P(e)\|$$

is injective for every $x \in V^{ab}$.

Automorphism group of the maximal abel cover

If X has no separating edges, then the automorphism group $\mathrm{Aut}(X^{ab})$ is a crystallographic group.

A group G is said to be a crystallographic group if it contains a subgroup H isomprphic to $\mathbb{Z}^d$ such that

(1) H is a normal subgroup, and G/H is finite, and

(2) H is the maximal abelian sungroup of G.

H is called the maximal lattice group.

$H_1(X, \mathbb{Z})$ is the maximal lattice in $\mathrm{Aut}(X^{ab})$, and the factor group $\mathrm{Aut}(X^{ab})/H_1(X, \mathbb{Z})$ is isomorphic to the automorphism group of X, $\mathrm{Aut}(X)$.

We thus have the exact sequence:

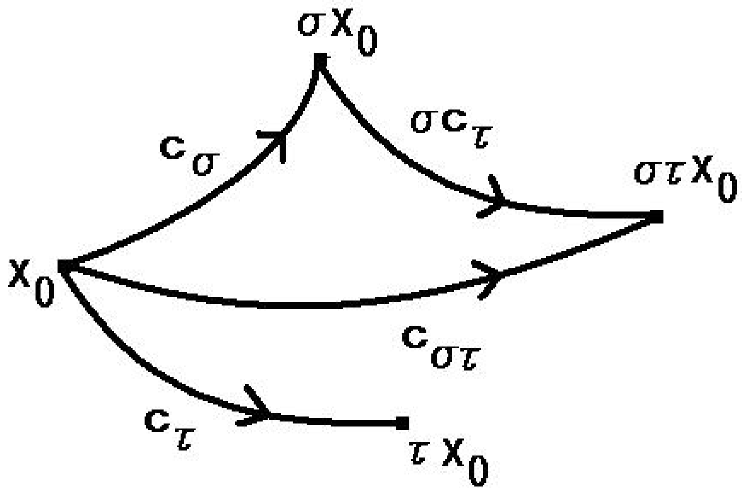
$$0 \to H_1(X, \mathbb{Z}) \to \mathrm{Aut}(X^{ab}) \to \mathrm{Aut}(X) \to 1.$$

In other words, $\mathrm{Aut}(X^{ab})$ is an extension of $H_1(X, \mathbb{Z})$ by the finite group $\mathrm{Aut}(X)$, and hence it is determined by a group cohomology class $\Theta \in H^2\big(\mathrm{Aut}(X), H_1(X, \mathbb{Z})\big)$.

More explicitly, Θ is represented by the 2-cocycle $\theta \in C^1\big(\mathrm{Aut}(X), H_1(X,\mathbb{Z})\big)$ defined by

$$\theta(\sigma,\tau) = [c_\sigma \cdot \sigma c_\tau \cdot \overline{c_{\sigma\tau}}], \quad (\sigma,\tau \in \mathrm{Aut}(X))$$

where c_σ is a path in X joining $x_0, \sigma x_0$.



II. The K_4 crystal –Diamond Twin–

How special are diamond and K_4 crystals ?

The diamond and K_4 crystals have maximal symmetry and the strong isotropic property.

Theorem **A crystal in $\mathbb{R}^3$ with maximal symmetry and strong isotropic property is either the diamond crystal or the K_4 crystal (and its mirror image).**

Thus the K_4 crystal is entitled to be called a diamond twin.

Remark **The K_4 crystal has chirality.**

Crystals

A crystal in physical sense is a periodic arrangement of atoms together with the bonding (depicted usually by virtual lines) of atoms provoked by atomic force.

A crystal in mathematical sense is a 1 dimensional figure in space with periodicity with respect to a lattice group action by translations (thus ignoring the physical characters of atoms and atomic forces in a crystal which may be different one by one).

Crystals with maximal symmetry

Definition **A crystal is said to have maximal symmetry if every automorphism of the crystal as an abstract graph extends to a congruent transformation.**

Note that every congruent transformation leaving the crystal invariant induces an automorphism, but not vice versa.

Strong isotropic property

Definition A crystal is said to have **strong isotropic property** if it is **regular**, say, of degree n, and if, for edges $e_1, \cdots, e_n$ with the same origin and $f_1, \cdots, f_n$ with another same origin, there exists a congruent transformation T such that $T(e_i) = f_i$, whatever order of edges you choose.

History of (Re)Discovery of the K_4 crystal

Chemical crystallographers have been trying to list (hypothetical) crystals with small "unit" set of vertices and small valence (degree).

- **It is believed that the crystallographer who discovered the crystal structure for the first time is Laves (1922).**
- **The crystal structure was called "(10,3)-a" by A. F. Wells (unclear whether he knew Laves's work).**

  A. F. Wells, Acta Cryst. 7 (1954), 535

  A. F. Wells, *Three Dimensional Nets and Polyhedra*, **Wiley (1977).**
- **H. S. M. Coxeter called it "Laves' graph of girth ten" (1955).**
- **M. O'Keeffe and his colleagues discussed this structure**

in some details and renamed it "srs" (2003). The structure is realized as the silicon net in a compound of silicon and strontium.

- **L. Danzer discovered the structure once again around 1994.**

- **It was pinned down by Sunada in the study of random walks on topological crystals.**

T. Sunada, Notices of the AMS, 55, 208 (2008).

I called it the K_4 crystal due to its mathematical relevance.

- **J. H. Conway calls the structure the "trimond" in his book published in 2008.**

The K_4 crystal is still "hypothetical" as a crystal composed of homo nuclei.

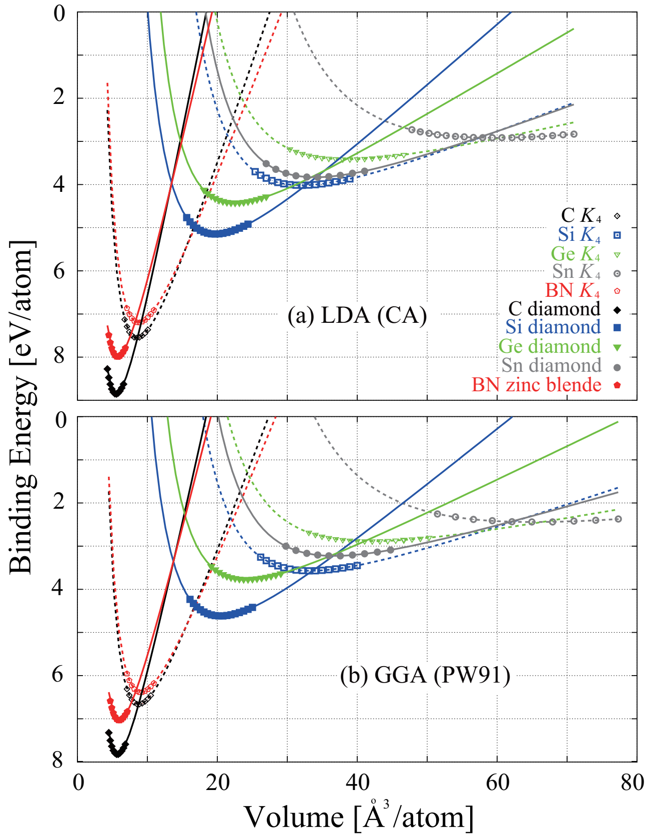
Possible physical properties of the K_4 crystal

Computations by a supercomputor are ongoing.

Masahiro Itoh, Tadafumi Adschiri, Yoshiyuki Kawazoe in Tohoku University.

C: Carbon
Si: Silicon
BN: Boron-Nitrogen
Ge: Germanium
Sn: Tin

The results imply the possibility of existence for the K_4 crystal structure as meta-stable ones for all cases.


Binding Energy [eV/atom]
Volume [Å^3/atom]
(a) LDA (CA)
(b) GGA (PW91)
C K_4
Si K_4
Ge K_4
Sn K_4
BN K_4
C diamond
Si diamond
Ge diamond
Sn diamond
BN zinc blende
0
2
4
6
8
0
20
40
60
80

II. Standard realizations

Topological Crystals

Consider now crystals of general dimension.

Definition **A d-dimensional topological crystal (or crystal lattice) is a regular covering graph over a finite graph, say X_0, with covering transformation group L isomorphic to $\mathbb{Z}^d$. L is said to be a periodic lattice.**

A periodic realization of X is a piecewise linear map $\Phi : X \longrightarrow \mathbb{R}^d$ such that there exists an injective homomorphism $\rho : L \longrightarrow \mathbb{R}^d$ satisfying

(1) $\Phi(\sigma x) = \Phi(x) + \rho(\sigma)$,

(2) $\rho(L)$ is a lattice group in $\mathbb{R}^d$.

The image $\Phi(X)$ is regarded as a "crystal" in $\mathbb{R}^d$.

Standard realizations

Among periodic realizations of a topological crystal, there is a "standard" one, which we shall call the standard realizations.

The standard realization is a generalization of $\widetilde{\Phi}$ for the Albanese map.

The canonical homomorphism $\mu : H_1(X_0, \mathbb{Z}) \longrightarrow L$ induces a linear map $\mu_{\mathbb{R}} : H_1(X_0, \mathbb{R}) \longrightarrow L \otimes \mathbb{R}$.

Equip $L \otimes \mathbb{R}$ with the inner product induced from the one on $H_1(X_0, \mathbb{R})$.

Define $\mathrm{v} : C_1(X_0, \mathbb{R}) \longrightarrow L \otimes \mathbb{R}$ by $\mathrm{v} = \mu_{\mathbb{R}} \circ P$ (remember that $P : C_1(X_0, \mathbb{R}) \longrightarrow H_1(X_0, \mathbb{R})$ is the projection).

Let $\widetilde{c} = (\widetilde{e_1}, \ldots, \widetilde{e_n})$ be a path in X with $o(\widetilde{c}) = x_0, t(\widetilde{c}) = x$. Then put $\widetilde{\Phi}(x_0) = 0$ and

$$\widetilde{\Phi}(x) = \mathrm{v}(e_1 + \cdots + e_n) = \mathrm{v}(e_1) + \cdots + \mathrm{v}(e_n),$$

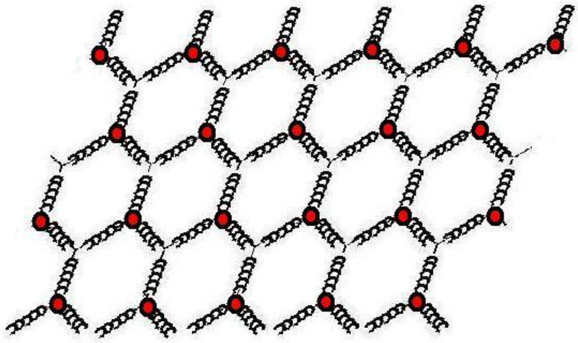
where $e_i \in E$ is the image of $\widetilde{e_i}$ by the covering map.

$\widetilde{\Phi} : X \longrightarrow L \otimes \mathbb{R} = \mathbb{R}^d$ is a periodic realization which we call the standard realization.

Energy minimizing property

The standard realization is characterized by energy minimizing property.

Think a crystal as a system of harmonic oscillators.



Each edge represents a harmonic oscillator whose energy is defined to be the square of its length.

We shall define the energy of a crystal "per a unit cell".

1. Given a bounded domain D in $\mathbb{R}^d$, denote by $\mathcal{E}(D)$ the sum of the energy of harmonic oscillators whose end points are in D, and normalize it in such a way as

$$\mathcal{E}_0(D) = \frac{\mathcal{E}(D)}{\deg(D)^{1-2/d}\mathrm{vol}(D)^{2/d}},$$

where $\deg(D)$ is the sum of degree of vertices in D.

2. **Take an increasing sequence of bounded domains $\{D_i\}_{i=1}^{\infty}$ with $\cup_{i=1}^{\infty} D_i = \mathbb{R}^d$ (for example, a family of concentric balls).**

The energy of the crystal is defined as the limit

$$E = \lim_{i\to\infty} \mathcal{E}_0(D_i).$$

- **The limit exists under a mild condition on $\{D_i\}_{i=1}^{\infty}$, and E does not depend on the choice of $\{D_i\}_{i=1}^{\infty}$.**
- **E is invariant under any homothetic transformation.**

Theorems

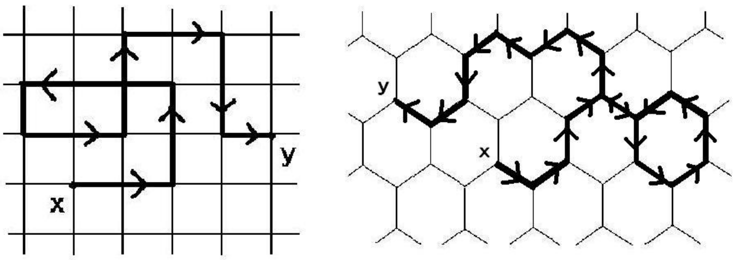
For a fixed topological crystal, the minimum of E is attained by a periodic realization, which coincides with the standard realization.

The standard realization has maximal symmetry.

Why the standard realization has maximal symmetry

The standard realization has a close relation with asymptotic behavior of random walks on a topological crystal.

"A random walker can detect the most natural way for the topological crystal to sit in space".



Let $p(n, x, y)$ be the nth step transition probability for the simple randam walk on a topological crystal X.

Simple random walk

A simple random walk on a graph in general is the random walk such that a particle moves in equal probability to a nearest neighbor along an edge.

$p(n, x, y)$ is the probability that a particle starting to move from x is found at y after n step movement.

Note that $p(n, x, y)$ is determined by the graph structure of X thus having nothing to do with its realization.

Relation between transition probabilities and standard realizations

Let $\Phi : X \longrightarrow \mathbb{R}^d$ be the standard realization. There exists a positive constant C such that

$$C\|\Phi(x) - \Phi(y)\|^2 = \lim_{n\to\infty} 2n\Big\{\frac{p(n,x,x)}{p(n,y,x)} + \frac{p(n,y,y)}{p(n,x,y)} - 2\Big\}$$

∘ This is a consequence of the asymptotic expansion of $p(n,x,y)$ as n goes to infinity.

∘ This is used when we prove that the standard realization yields a realization with maximal symmetry.

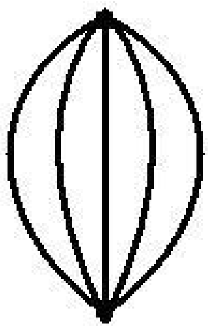
Problem

List all crystals in $\mathbb{R}^d$ with the strong isotropic property and maximal symmetry

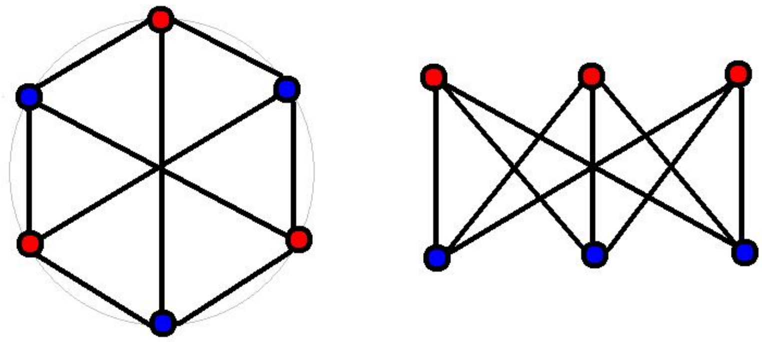
In every dimension, there is at least one crystal with these properties, that is, a generalization of the diamond crystal, the standard realization of the maximal abelian covering of the graph consisting of two vertices joined by $d+1$ multiple edges.

Examples of 4D strongly isotropic crystals

1) 4-dimensional diamond



2) The standard realization of the maximal abelian covering graph of the bipartite complete graph $K_{3,3}$



Problem **Is there a strongly isotropic crystal of degree 4 ?**