Large Deviation Asymptotics of Heat Kernels on Periodic Manifolds

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Main Topic to be treated

Establish a long-time asymptotic of the heat kernel on a periodic manifold

The heat kernel $k(t,x,y)$ on a Riemannian manifold is the fundamental solution of the heat equation

$$\frac{\partial u}{\partial t} + \Delta u = 0, \quad u(t,x)\big|_{t=0} = f(x),$$

$$u(t,x) = \int_X k(t,x,y)f(x)dx$$

∘ **Key Words**:

periodic manifolds (infinite-fold abelian covers over closed manifolds)

Albanese maps (a special harmonic map of a periodic manifold into a Euclidean space)

large deviation (a technical term in probability)

The heat kernel on $\mathbb{R}^n$

$$k(t, x, y) = (4\pi t)^{-n/2} \exp\Big(- \frac{\|x - y\|^2}{4t} \Big)$$

General Problen: What is the shape of the heat kernel on a general open manifold ?

Of course, it is impossible to give an exact shape in general.

There are three typical ways to explore the shape.

1. Estimates (from above and below) by a "Gaussian" function.

Naive Gaussian function :

$$(4\pi t)^{-m/2} \exp\left(-\frac{d(x,y)^2}{4t}\right)$$

where $m = \dim X$ and $d(x,y)$ denotes the Riemannian distance between x and y.

In general, it must be replaced by more sophisticated functions.

2. Asymptotics at $t = 0$: Local nature

$$k(t,x,y) \sim (4\pi t)^{-m/2} \exp\left(-\frac{d(x,y)^2}{4t}\right) \times \left(a_0(x,y) + a_1(x,y)t + a_2(x,y)t^2 + \dots\right) \quad (t \downarrow 0),$$

The nature of coefficients $a_i(x, y)$ is "local" in the sense that they are described by quantities defined only on a neighborhood of the shortest geodesic joining x and y.

This is roughly explained by the intuitive observation that the short time behavior of the heat diffusion on X should be similar to the one on the Euclidean space.

3. Asymptotics at $t = \infty$: Global nature

Our interest is in asymptotics at $t = \infty$.

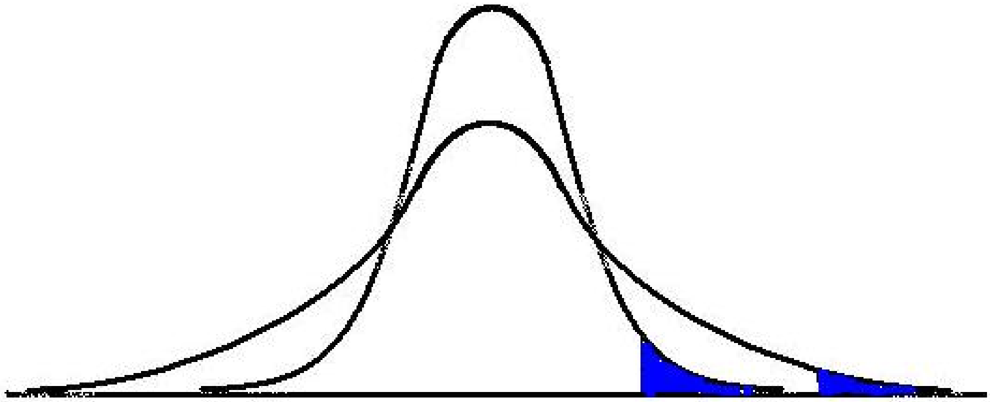
There are at least two kinds of asymptotics at $t = \infty$

(1) Central-limit-theorem type

(2) Large-diviation type

General Philosophy of Large Deviation Theory: It, in general, concerns the asymptotic behavior of remote tails of sequences of probability distributions

Remote tails



Example: For the heat kernel on R^n,

(1) Central-limit-theorem When $y_t - \sqrt{t}\xi$ is bounded,

$$k(t,x,y_t) \sim (4\pi t)^{-n/2} \exp(-\|\xi\|^2/4) \quad (t \to \infty)$$

(2) Large-diviation When $\lim_{t\to\infty}(y_t - t\xi) = \mathrm{a}$,

$$k(t,x,y_t) \sim (4\pi t)^{-n/2} \exp(-t\|\xi\|^2/4) \times \exp\big(\xi \cdot (x - \mathrm{a})/2\big) \quad (t \to \infty)$$

How to formulate these asymptotics for more general Riemannian manifolds.

If one wants to establish asymptotics similar to the case of $\mathbb{R}^n$, the following questions come up:

(1) Where does a vector $\boldsymbol{\xi}$ (and a vector a) live ?

(2) How do $y_t - \sqrt{t}\xi$ and $y_t - t\xi$ make sense ?

The asymptotics mentioned above are generalized to the case of periodic manifolds.

> **Theorem** Let X be a periodic manifold, and $k(t,x,y)$ be the heat kernel on X. Let $\Phi : X \longrightarrow \mathbb{R}^d$ be the Albanese map.
>
> (1) Central-limit-theorem When $\Phi(y_t) - \sqrt{t}\xi$ is bounded,
>
> $$k(t,x,y_t) \sim C_1(4\pi t)^{-d/2}\exp(-C_2\|\xi\|^2/4) \quad (t \to \infty)$$
>
> with positive geometric constants C_1, C_2.

Note the exponent of $(4\pi t)^{-d/2}$ is different from the one for $(4\pi t)^{-m/2}$ in the short time asymptotic.

(2) Large-diviation When $\lim_{t\to\infty}(\Phi(y_t) - t\xi) = a$,

$$\begin{aligned} & k(t,x,y_t) \\ \sim\ & C(4\pi t)^{-d/2}\exp(-tH(\xi))f\big(\pi(x)\big)g\big(\pi(y_t)\big) \\ & \qquad\qquad \times\exp\big(\omega_0\cdot\big(\Phi(x)-a\big)\big) \\ & \qquad\qquad\qquad (t\to\infty) \end{aligned}$$

with a positive constant C and a positive-valued convex function $H(\xi)$

The terms $d, \Phi, \pi, f, g, \omega_0$ are explained later.

A weak version For a sequence $\{y_t\}_{t>0}$ in X, we have

$$\lim_{t\to\infty} \frac{1}{t} \log k(t, x, y_t) = -H(\xi)$$

provided that $\|\Phi(y_t) - t\xi\|$ is bounded.

The function H is an analogue of the entropy in thermodynamics.

A periodic manifold X is an abelian covering manifold of a closed Riemannian manifold, say M, with free abelian covering transformation group.

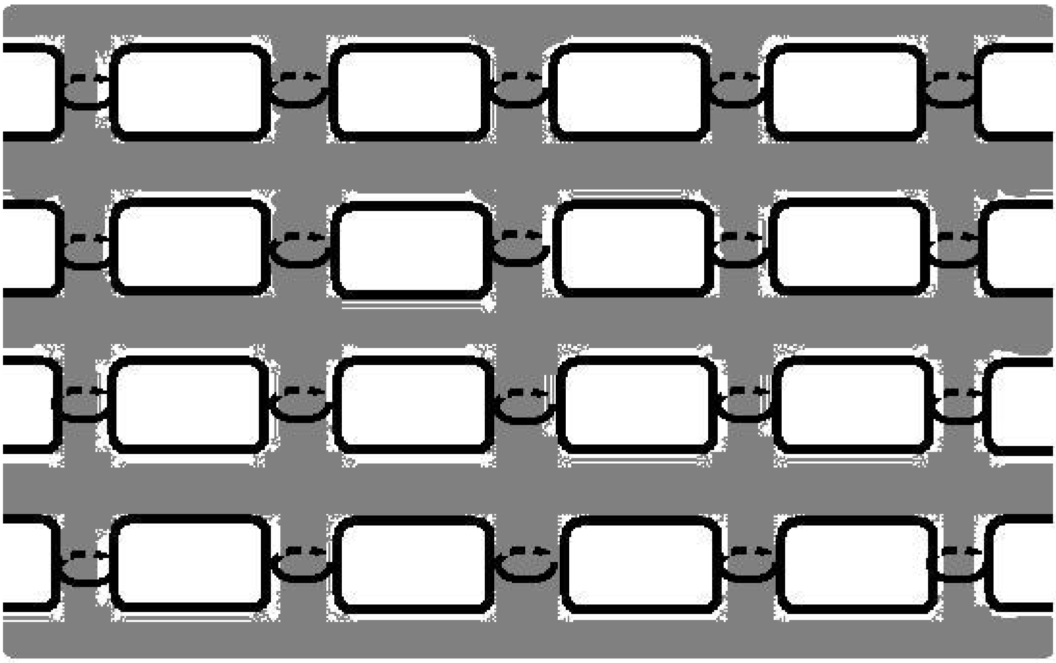
- **$\pi : X \longrightarrow M$ is the covering map.**
- **d is the rank of the covering transformation group, Γ ($\Gamma = \mathbf{Z}^d$).**

(Remark: In general, $d \neq \dim X$)

Example (1) $\mathbf{R}^n$. This is the $\mathbf{Z}^n$-covering over the flat torus $\mathbf{R}^n/\mathbf{Z}^n$.

(2) The homology universal covering manifold (the maximal abelian covering manifold) of a closed manifold M. This is an abelain covering manifold with the covering transfromation group $H_1(M, \mathbf{Z})$.

(3) 2-dimensional periodic manifold given the the following figure.



X is an abelian covering of $M \Longrightarrow$

- The surjective homomorphism

$$\rho : H_1(M, \mathrm{Z}) \longrightarrow \Gamma \to 0$$

- The injective linear map

$${}^t\rho_{\mathrm{R}} : \mathrm{Hom}(\Gamma, \mathrm{R}) \longrightarrow H^1(M, \mathrm{R})$$

$H^1(M, \mathrm{R})$ is identified with the space of harmonic 1-forms, and equipped with the Hodge metric (inner product) defined by

$$\omega \cdot \omega = \int_M |\omega|^2$$

In this way, we equip $\mathrm{Hom}(\Gamma, \mathrm{R})$ (also its dual $\Gamma \otimes \mathbb{R}$) with an inner product.

The Albanese map

$$\Phi : X \longrightarrow \Gamma \otimes \mathrm{R} (= \mathbb{R}^d)$$

is defined by

$$\langle \omega, \Phi(x) \rangle = \int_{x_0}^{x} \widetilde{\omega}$$

where

$$\omega \in \mathrm{Hom}(\Gamma, \mathrm{R}) \subset H^1(M, \mathrm{R}),$$
$\widetilde{\omega}$ is the lifting of ω to X

(Note $(\Gamma \otimes \mathrm{R})^* = \mathrm{Hom}(\Gamma, \mathrm{R})$)

In the term $\Phi(y_t) - t\xi$ (or $\Phi(y_t) - \sqrt{t}\xi$), ξ is a vector in $\Gamma \otimes \mathrm{R}$.

To define $H(\xi)$, the following lemma is required.

Lemma

Let Δ_M be the analyst's Laplacian on M. For a vector field v and a function f, the operaotr $\Delta_M + v + f$ has a simple eigenvalue λ_0 with a positive valued eigenfunction.

λ_0 will be called the Perron-Frobenius eigenvalue (P-F eigenvalue).

For $\omega \in \mathrm{Hom}(\Gamma, \mathrm{R})$, define the operator D_ω by

$$D_\omega f = \Delta_M f + 2\langle \omega, df \rangle + |\omega|^2 f$$

and let $\lambda_0(\omega)$ be the P-F eigenvalue of D_ω.

- $\lambda_0(-\omega) = \lambda_0(\omega)$

Lemma λ_0 **is an analytic function on** $\mathrm{Hom}(\Gamma, \mathrm{R})$ **with**

$$\mathrm{Hess}\ \lambda_0 > 0 \quad \text{(everywhere)}$$

Define the gradient map

$$\nabla\lambda_0 : \mathrm{Hom}(\Gamma, \mathrm{R}) \longrightarrow \Gamma \otimes \mathrm{R}$$

by

$$(\nabla_v \lambda_0)(\omega) = \frac{d}{dt}\Big|_{t=0} \lambda_0(\omega + tv)$$

Lemma $\nabla\lambda_0$ is a diffeomorphism.

The element ω_0 is defined as an element of $\mathrm{Hom}(\Gamma, \mathrm{R})$ with

$$\nabla\lambda_0(\omega_0) = \xi$$

The case of maximal abelian covering manifolds

λ_0 is a convex function on $H^1(M, \mathrm{R})$.

The gradient map $\nabla\lambda_0$ is a diffeomorphism of $H^1(M, \mathrm{R})$ onto $H_1(M, \mathrm{R})$.

Definition of $H(\xi)$

$$H(\xi) = \sup_{\omega}(\langle \xi, \omega \rangle - \lambda_0(\omega))$$

This is the Legendre-Fenchel transform.

Note that the supremum in the right hand side is attained by ω_0 with $\nabla \lambda_0(\omega_0) = \xi$, and hence

$$H(\xi) = \langle \xi, \omega_0 \rangle - \lambda_0(\omega_0)$$

The functions f, g on M are positive valued eigenfunctions of D_{ω_0} and $D_{-\omega_0}$ for the P-F eigenvalue $\lambda_0(\omega_0) = \lambda_0(-\omega_0)$:

$$D_{\omega_0} f = \lambda_0(\omega_0) f, \quad D_{-\omega_0} g = \lambda_0(-\omega_0) g$$

f, g are normalized as

$$\int_M fg = 1$$

Every terms in Theorem have been defined.

∘ Rough Idea

Theorem (1) is a direct consequence of the local central limit theorem.

$$\lim_{t\uparrow\infty}\Big((4\pi t)^{d/2}k(t,x,y) - C(X)\exp\big(-\frac{\mathrm{vol}(M)}{4t}\|\Phi(x)-\Phi(y)\|^2\big)\Big)=0,$$

uniformly for all $x, y \in X$.

M. Kotani and T. Sunada, *Albanese maps and off diagonal long time asymptotics for the heat kernel*, **Comm. Math. Phys., 209(2000), 633-670.**

The idea for Theorem (2).

◦ Define the function u on X by

$$u(x) = \langle \omega_0, \Phi(x) \rangle = \int_{x_0}^{x} \widetilde{\omega}_0.$$

◦ Let $\widetilde{D}_0$ be the lifting of $D_0 = D_{\omega_0}$ to X.

◦ Let $k_0(t, x, y)$ be the kernel function of $e^{t\widetilde{D}_0}$ on X. Since $\widetilde{D}_0 = e^{-u}\Delta_X e^u$, we have

$$\begin{aligned} k(t, x, y) &= k_0(t, x, y)\frac{e^{u(x)}}{e^{u(y)}} \\ &= k_0(t, x, y)\exp\langle \omega_0, (\Phi(x) - \Phi(x))\rangle \end{aligned}$$

∘ **Look at the direct integral decomposition**

$$\widetilde{D}_0 = \int_{\widehat{\Gamma}}^{\oplus} (D_0)_\chi \, d\chi$$

$(D_0)_\chi$ **is the operator, induced from** D_0**, acting in sections of the line bundle associated with** χ**.**

∘ **Identify the group of unitary characters** $\widehat{\Gamma}$ **with the torus** $\mathrm{Hom}(\Gamma, \mathrm{R})/\mathrm{Hom}(\Gamma, \mathrm{Z})$ **via the correspondence**

$$\omega \Longleftrightarrow \chi(\cdot) = \exp 2\pi\sqrt{-1} \int_{\cdot} \omega$$

∘ Let $k_\omega(t,p,q)$ be the kernel function of $\exp\left(tD_{\omega_0+2\pi\sqrt{-1}\omega}\right)$ on M. Observe that $D_{\omega_0+2\pi\sqrt{-1}\omega}$ is unitarily equivalent to $(D_0)_\chi$, and

$$k_0(t,x,y) = \int_{\mathrm{Hom}(\Gamma,\mathrm{R})/\mathrm{Hom}(\Gamma,\mathrm{Z})} k_\omega(t,\pi(x),\pi(y)) \exp\langle\omega,\Phi(x)-\Phi(y)\rangle d\omega$$

∘ $D_{\omega_0+2\pi\sqrt{-1}\omega}$ has a simple eigenvalue $\lambda_0(\omega_0+2\pi\sqrt{-1}\omega)$ as far as ω is in a small neiborhood $U(0)$ of 0.

- **We have the expression**

$$\exp\left(tD_{\omega_0+2\pi\sqrt{-1}\omega}\right) = e^{t\lambda_0(\omega_0+2\pi\sqrt{-1}\omega)}P_\omega + Q_\omega(t)$$

such that the kernel function of P_ω is $f_\omega(p)\overline{g_\omega(q)}$ where

$$D_{\omega_0+2\pi\sqrt{-1}\omega}f_\omega = \lambda_0(\omega_0+2\pi\sqrt{-1}\omega)f_\omega,$$
$${}^tD_{\omega_0+2\pi\sqrt{-1}\omega}g_\omega = \lambda_0(\omega_0+2\pi\sqrt{-1}\omega)g_\omega,$$
$$\int_M f_\omega\overline{g_\omega} = 1$$

and the kernel function $q(t,p,q)$ of $Q_\omega(t)$ satisfies

$$|q(t,p,q)| \leq e^{ct} \quad (0 < c < \lambda_0(\omega_0))$$

∘ **Put**

$$\varphi(\omega) = \lambda(\omega_0 + \omega) - \langle \xi, \omega_0 + \omega \rangle + H(\xi)$$

Note

$$\begin{aligned} &\varphi(0) = 0, \\ &\nabla \varphi(0) = 0, \\ &\mathrm{Hess}_0 \varphi = \mathrm{Hess}_{\omega_0} \lambda_0 > 0 \end{aligned}$$

∘ **We find**

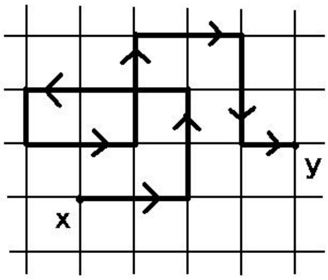
$$\begin{aligned} k_0(t, x, y_t) \ \sim \ & e^{-tH(\xi)} \int_{U(0)} \exp(t\varphi(2\pi\sqrt{-1}\omega)) \\ & \times \exp(t\langle \xi, \omega_0 + 2\pi\sqrt{-1}\omega \rangle) \\ & \times f_\omega(\pi(x)) \overline{g_\omega(\pi(y_t))} \\ & \times \exp(\langle \omega, \Phi(x) - \Phi(y) \rangle) d\omega \end{aligned}$$

Apply the Laplace method to obtain Theorem (2).

Problem: Formulate large deviation asymptotics in the case of negatively curved spaces.

Discrete Analogue

A similar idea may be applied to the asymptotics of the transition probability of random walks on a crystal lattices, an abelian covering graph of finite graphs with a free abelian covering transformation group.



Replacement

$X \Longrightarrow$ crystal lattice,

$k(t,x,y) \Longrightarrow p(n,x,y)$ (n-step transition probability),

(Brownian motion $\Longrightarrow$ (simple) randoma walk),

$e^{t\Delta} \Longrightarrow L^n$ $(n = 0, 1, 2, \ldots)$,

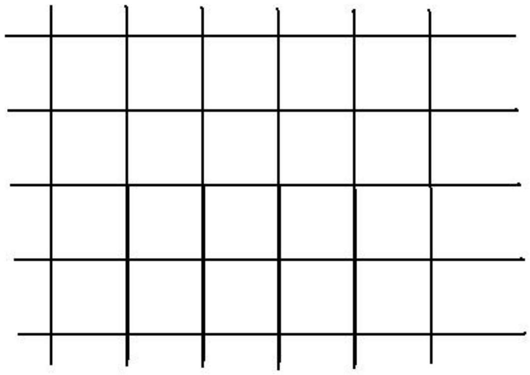
$\lambda_0 \Longrightarrow \log \mu_0$,

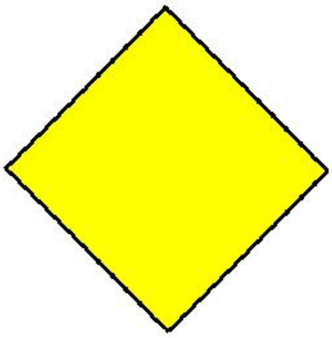
$\Phi \Longrightarrow$ standard realization

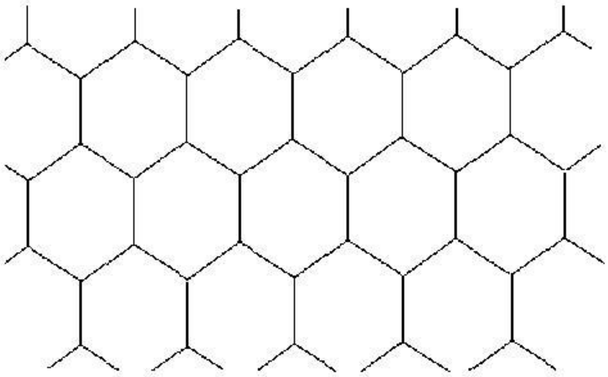
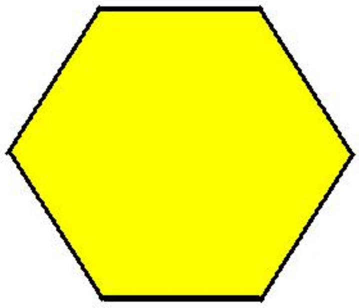
where L is the transition operator, and μ_0 is the maximal positive eigenvalue.

The formulation of theorems and proofs can be done in a parallel way as the continuous case.

M. Kotani and T. Sunada, *Large deviation and the tangent cone at infinity of a crystal lattice*, Math. Z., 254 (2006), 837-870.





For the large deviation asymptotics, a difference is in the fact that Image($\nabla\lambda_0$) is the interior of a convex polyhedron D in $\Gamma \otimes \mathrm{R}$ (a consequence of "finite propagation speed" for random walks).

In the case of the maximal abelian covering graphs over a finite graph X_0, the image of the gradient map $\nabla\lambda_0 : H^1(X_0, \mathrm{R}) \longrightarrow H_1(X_0, \mathrm{R})$ coincides with the unit ball in

$H_1(X_0, \mathrm{R})$ with respect to the ℓ^1-norm.

Given $\xi \in \mathrm{Int}\ D$, we have a large deviation asymptotic for $p(n, x, y_n)$ for y_n with $\lim_{n\to\infty}(\Phi(y_n) - n\xi) = a$.

A weak version of large deviation asymptotics:
If $\|\Phi(y_n) - n\xi\|$ is bounded, then

$$\lim_{n\to\infty} \frac{1}{n} \log p(n, x, y_n) = -H(\xi).$$

Open problem

What about for $\xi \in \partial D$?