Radial Basis Functions: An Introduction and Potential Applications to P.D.E.s

A radial basis function is simply a function $\phi : R^n \to R$ which is radial and positive definite (or, more generally, conditionally positive definite). Under mild assumptions such as the Fourier transform $\hat{f} > 0$ on R^n, linear combinations of shifts of these functions can always interpolate scattered data on R^n. That is, given a (finite) point set $X \subset R^n$ and a continuous $f : R^n \to R$, there is a unique interpolant $I_X(f)$ from

$$V_X := span\{\phi(x - x_j) : x_j \in X\}$$

which agrees with f on X. More generally such interpolants can be constructed which interpolate a (properly smooth) function on any set of linear functionals which are linearly independent on the space of test functions, for example (partial) derivatives of point evaluation functionals. Such flexibility gives RBFs good potential for use in collocation methods. Recently good error estimates have been obtained when interpolating functions $f \in W_2^k(\Omega)$, where $\Omega \subset R^n$ is compact with Lipschitz boundary.

In another direction, matrix-valued RBFs have been constructed which can be used to interpolate vector fields (i.e. functions $g : R^2 \to R^2$ or $h : R^3 \to R^3$). Moreover these interpolants can preserve certain properties of the underlying vector field such as being divergence or curl free. Such ideas are relevant to fluid-flow problems and problems in electricity and magnetism.