A Darcy-Forchheimer model

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This is common work with M. Wheeler, I.C.E.S., University of Texas at Austin.

We consider the steady Darcy-Forchheimer flow of a single-phase fluid in a porous medium in a two or three dimensional domain Ω with boundary $\partial\Omega$:

$$\frac{\mu}{\varrho} K^{-1} \boldsymbol{u} + \frac{\beta}{\varrho} |\boldsymbol{u}| \boldsymbol{u} + \nabla p = \boldsymbol{0} \text{ in } \Omega \,,$$

$$\operatorname{div} \boldsymbol{u} = b \text{ in } \Omega \,,$$

$$\boldsymbol{u} \cdot \boldsymbol{n} = g \text{ on } \partial\Omega \,,$$

where ϱ is the density of the fluid, μ its viscosity, β a dynamic viscosity, all assumed to be positive constants, K is the permeability tensor, assumed to be uniformly positive definite and bounded, and b and g are given functions satisfying the compatibility condition:

$$\int_\Omega b(\boldsymbol{x}) d\boldsymbol{x} = \int_{\partial\Omega} g(\sigma) d\sigma \,.$$

This nonlinear problem is of monotone type. Under mild regularity assumptions on the data b and g, several authors have proven that it has a unique weak solution. We propose to solve it numerically with a finite-element method: discontinuous $\mathbb{P}_0^d$ elements for the velocity $\boldsymbol{u}_h$, $d = 2$ or 3, and discontinuous $\mathbb{P}_1$ Crouzeix-Raviart elements for the pressure p_h:

$$\forall \boldsymbol{v}_h \,, \ \frac{\mu}{\varrho} \int_\Omega (K^{-1} \boldsymbol{u}_h) \cdot \boldsymbol{v}_h \, d\boldsymbol{x} + \frac{\beta}{\varrho} \int_\Omega |\boldsymbol{u}_h| \boldsymbol{u}_h \cdot \boldsymbol{v}_h \, d\boldsymbol{x} + \sum_T \int_T \nabla p_h \cdot \boldsymbol{v}_h \, d\boldsymbol{x} = 0 \,,$$

$$\forall q_h \,, \ \sum_T \int_T \nabla q_h \cdot \boldsymbol{u}_h \, d\boldsymbol{x} = - \int_\Omega q_h \, b \, d\boldsymbol{x} + \int_{\partial\Omega} q_h \, g \, d\sigma \,.$$

We prove that this scheme is convergent, again under mild regularity assumptions on the data, and of order one if the exact solution is sufficiently smooth. This non-linear scheme can be solved by a convergent alternating-directions algorithm.