

Algebra Qualifying Examination

January 2002

Directions:

1. Answer all questions. (Total possible is 100 points.)
2. Start each question on a new sheet of paper.
3. Write only on one side of each sheet of paper.

Policy on Misprints:

The Qualifying Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do not interpret the problem in such a way that it becomes trivial.

Notes:

Each ring R referred to in this exam will be assumed to contain a multiplicative identity 1_R . Every left R -module M will have the property that $1_R \cdot m = m$ for all $m \in M$, and every homomorphism of rings $\varphi: R \rightarrow S$ will be assumed to satisfy the property that $\varphi(1_R) = 1_S$. \mathbb{Q} is the field of rational numbers, \mathbb{R} the field of real numbers, \mathbb{C} the field of complex numbers, \mathbb{Z} the ring of integers, and \mathbb{F}_q a finite field with q elements.

1. (14pts) Let G be a group of order 30.
 - a) Prove that either G has a normal subgroup K of order 5 or a normal subgroup H of order 3, and conclude that G contains a normal subgroup that is cyclic of order 15.
 - b) Show that up to isomorphism there are exactly four distinct groups of order 30. (They are \mathbb{Z}_{30} , D_{15} , $\mathbb{Z}_3 \times D_5$, and $\mathbb{Z}_5 \times S_3$.)
2. (12pts) Let G be a group and let $\text{Aut}(G)$ denote the group of automorphisms of G . Recall $\varphi \in \text{Aut}(G)$ is called an inner automorphism if there exists an element $g \in G$ such that $\varphi(x) = gxg^{-1}$ for all $x \in G$. Let $\text{Inn}(G)$ denote the set of all inner automorphisms of G .
 - a) Prove that $\text{Inn}(G)$ is a normal subgroup of $\text{Aut}(G)$.
 - b) Show that $\text{Inn}(G)$ is isomorphic to the quotient of G by its center $C(G)$.
 - c) Give an example of a finite group G and an automorphism φ that is *not* inner.
3. (12pts) Let R be a commutative ring (with $1_R \neq 0_R \in R$) and let $I \subsetneq R$ be a proper ideal.
 - a) Show that there is a natural one-to-one correspondence between the prime ideals of R/I and the prime ideals of R that contain I .
 - b) Show that there exists a prime ideal P with $I \subset P \subset R$ which is minimal in the sense that if P' is another prime ideal of R with $I \subset P' \subset P \subset R$ then $P' = P$.

4. (14pts) Let R be a commutative ring with $1_R \in R$. An R -module I is said to be injective if, given any diagram of R -module homomorphisms

$$\begin{array}{ccc} 0 & \longrightarrow & A \xrightarrow{g} B \\ & & f \downarrow \\ & & I \end{array}$$

with the top row exact (i.e. g a monomorphism), there exists an R -module homomorphism $h: B \rightarrow I$ such that $h \circ g = f$

$$\begin{array}{ccc} 0 & \longrightarrow & A \xrightarrow{g} B \\ & & f \downarrow \quad \swarrow h \\ & & I \end{array}$$

It can be shown that every R -module may be embedded in an injective R -module.

- a) Show that the following are equivalent:
- 1) I is injective.
 - 2) every short exact sequence $0 \longrightarrow I \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is split exact (i.e. $B \cong I \oplus C$ in such a way that f and g become the canonical inclusion and projection respectively.)
 - 3) I is a direct summand of any module B of which it is a submodule.
- b) Show that I is injective if and only if for every ideal $J \subset R$, any R -module homomorphism $J \rightarrow I$ may be extended to an R -module homomorphism $R \rightarrow I$.
5. (12pts) Let $\overline{\mathbb{Q}} = \{z \in \mathbb{C} \text{ s.t. } \exists \text{ a polynomial with rational coefficients, } F(X) \in \mathbb{Q}[X], \text{ of degree } \geq 1 \text{ with } F(z) = 0\}$.
- a) Prove that $\overline{\mathbb{Q}}$ is a field.
 - b) Prove that $\overline{\mathbb{Q}}$ is algebraically closed.
6. (12pts) Consider the polynomial $X^5 - X + 1 \in \mathbb{F}_5[X]$. Show that $F(X)$ is irreducible and determine the splitting field K over \mathbb{F}_5 of $F(X)$. Determine the Galois group $\text{Gal}(K/\mathbb{F}_5)$ and explain how the Frobenius automorphism $\varphi_5: K \rightarrow K$ defined by $\varphi_5(x) = x^5$ in $\text{Gal}(K/\mathbb{F}_5)$ permutes the roots of $F(X)$.
7. (12pts) Let $T: V \rightarrow V$ be a symmetric operator on a real vector space V with an inner product, i.e. a positive definite symmetric bilinear form $\langle \cdot, \cdot \rangle: V \times V \rightarrow \mathbb{R}$. Assume V is finite dimensional. Show that there is an orthonormal basis of V consisting of eigenvectors of T . (Recall a symmetric operator satisfies $\langle v, Tw \rangle = \langle Tv, w \rangle$ for all $v, w \in V$.)
8. (12pts) Let R be a commutative ring with $1_R \in R$ and M a Noetherian R -module (i.e. every submodule of M is finitely generated, including M itself.) Suppose $f: M \rightarrow M$ is a surjective homomorphism, prove that f is also injective. (Hint: Examine the kernels of the iterates, $f^n: M \rightarrow M$, of f .)