Algebra Qualifying Examination
January, 2003

Directions:

1. Answer all questions. (Total possible is 100 points.)
2. Start each question on a new sheet of paper.
3. Write only on one side of each sheet of paper.

Policy on Misprints:

The Qualifying Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do not interpret the problem in such a way that it becomes trivial.

Notes:

1. All rings are unitary. All modules are unitary.
2. $\mathbb{Q}$ is the rationals, $\mathbb{R}$ the reals, $\mathbb{C}$ the complexes, and $\mathbb{Z}$ the integers.

Problems

1. (10 points) Let $T\colon\ \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation. Prove that T has a one dimensional invariant subspace and a two dimensional invariant subspace.
2. (10 points) Let $G = \mathbb{Q}/\mathbb{Z}$. Prove that, for all $t \in \mathbb{Z}^+, G$ has a unique cyclic subgroup of order t.
3. (10 points) Let F_p be the finite field with p elements, p a prime. Let $f(x) \in F_p[x]$ be an irreducible polynomial of degree $n \geq 2$. Let a, b be roots of $f(x)$ in some extension field K of F_p.
   a. Prove that a and b have the same order in the multiplicative group K^* of nonzero elements of K.
   b. Show that this order is equal to the least positive integer s such that $f(x)$ divides $x^s - 1$.
4. (10 points) Let G be a group. Let X be a nonempty set.
   (i) (a) Define what it means for G to act on X.
   (b) If $x \in X$, define the orbit of x under the action of G on X.
   (ii) (a) S_5 acts on itself by conjugation. List one element from each orbit.
   (b) How many elements are in the orbit containing (12)(34)?
   (c) How many elements in S_5 commute with (12)(34)?

5. (10 points) Let M and N be (left) modules over the ring R. Let $f\colon\ M \to N$ be an R-module homomorphism. Let K be the kernel of f, and let L be a submodule of K. Prove that f induces an R-module homomorphism of M/L into N.

6. (10 points) Suppose G is a finite group. Let p be a prime. Suppose H is a normal p-subgroup of G. Prove the following.

   a. H is contained in each Sylow p-subgroup of G.

   b. If K is a normal p-subgroup of G, then HK is a normal p-subgroup of G.

7. (10 points) Let R be the subring of $\mathbb{Q}[x]$ consisting of all polynomials in $\mathbb{Q}[x]$ that have integer constant terms. Then R is easily seen to be an integral domain with units ± 1.

   (a) Show that x is not an irreducible element of R.

   (b) Show that x cannot be written as a product of irreducibles and conclude that R is not a unique factorization domain.

8. (10 points) Let F and L be fields with $F \subseteq L \subseteq \mathbb{C}$. Let $f(x) \in F[x]$ and $g(x) \in L[x]$ with $f(x)$ irreducible over F. Suppose $c \in \mathbb{C}$ is a root of both $f(x)$ and $g(x)$ and c is the only common root of $f(x)$ and $g(x)$ in $\mathbb{C}$. Prove that c is in L.

9. (10 points) Recall that a (left) R-module M is *simple* (or *irreducible*) if $M \neq \{0\}$ and M has no proper nonzero submodules. Prove that if M is simple then $M \cong R/L$ where L is a maximal left ideal of R.

10. (10 points) Compute the Galois group of $x^{10} - 1 \in \mathbb{Q}[x]$ and justify your answer.