Algebra Qualifying Examination
January, 2004

Directions:

1. Answer all questions. (Total possible is 100 points.)
2. Start each question on a new sheet of paper.
3. Write only on one side of each sheet of paper.

Policy on Misprints:

The Qualifying Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do not interpret the problem in such a way that it becomes trivial.

Notes:

1. All rings are unitary. All modules are unitary.
2. $\mathbb{Q}$ is the rationals, $\mathbb{R}$ the reals, $\mathbb{C}$ the complexes, and $\mathbb{Z}$ the integers.

Questions

1. (10 points) Determine all groups that have exactly 3 subgroups.
2. (15 points) Let $Z(G)$ and $\mathrm{Inn}(G)$ denote the center of the group G and the group of inner automorphisms of G respectively.
   (i) Prove that $G/Z(G)$ is isomorphic to $\mathrm{Inn}(G)$.
   (ii) Suppose that the group $\mathrm{Aut}(G)$ of automorphisms of G is cyclic. Prove that G is abelian.
3. (10 points) Let V be a vector space of dimension n over a field F. Let $T\colon\ V \to V$ be a linear transformation. A vector $v \in V$ is called a cyclic vector with respect to T if $\{v, Tv, T^2v, \ldots, T^{n-1}v\}$ is a basis for V.
   (i) Prove that if V has a cyclic vector with respect to T then the minimal polynomial for T, $m(x) \in F[x]$, has degree n.
   (ii) Write the statement that is the converse of (i). Is this statement true? No proof required.
4. (10 points) Let R be a commutative ring with 1. Recall that R is Noetherian if it satisfies the ascending chain condition (ACC) on ideals.
   Prove that R is Noetherian if and only if every ideal of R is finitely generated.

5. (10 points) Let R be a ring with 1. Let M a left unital R-module. Let S and T be submodules of M. Prove that the following are equivalent.

   (i) M is isomorphic to the external direct sum of S and T using $(s, t) \to s + t$.

   (ii) $S + T = M$ and $S \cap T = \{0\}$.

   (iii) Each $m \in M$ has a unique expression of the form $m = s + t$ for $s \in S$ and $t \in T$.

6. (10 points)

   (i) If $R_1, \ldots, R_n$ are rings with identity and I is an ideal in $R_1 \oplus \cdots \oplus R_n$ then $I = A_1 \oplus \cdots \oplus A_n$, where each A_i is an ideal in R_i.

   (ii) Show that the conclusion in (i) does not need to hold if the rings R_i do not have identities.

7. (10 points) The ring $\mathbb{Z}[i]/(2+i)$ is isomorphic to the field $\mathbb{Z}_5$. Exhibit a ring homomorphism ϕ from $\mathbb{Z}[i]$ onto $\mathbb{Z}_5$ whose kernel is the ideal $(2+i)$. Show that ϕ is a homomorphism and that its kernel is $(2+i)$.

8. (15 points) Let c be a primitive fifth root of unity.

   (i) Explain why $\mathbb{Q}(c)$ is a Galois extension of degree 4.

   (ii) Prove that $\mathbb{Q}(c)$ has a subfield K of degree 2 over Q and it is unique.

   (iii) Prove that $K = Q(\sqrt{5})$.

9. (10 points) Prove that there exists a field of order p^n for each prime p and each positive integer n. You may use the fact that $\mathbb{Z}_p$ is a field of order p.